Central Limit Theorem for Linear Statistics of Eigenvalues of Band Random Matrices

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Abstract

We prove the Central Limit Theorem for linear statistics of the eigenvalues of band random matrices provided $\sqrt{n} \ll b_n \ll n$ and test functions are sufficiently smooth.

1 Introduction

The goal of this paper is to prove the Central Limit Theorem for linear statistics of the eigenvalues of real symmetric band random matrices with independent entries.

First, we define a real symmetric band random matrix. Let $\{b_n\}$ be a sequence of integers satisfying $0 \le b_n \le n/2$ such that $b_n \to \infty$ as $n \to \infty$. Define

$$d_n(j,k) := \min\{|k-j|, n-|k-j|\}, \tag{1.1}$$

$$I_n := \{(j,k) : d_n(j,k) \le b_n,\ j,k = 1, ..., n\}, \quad \text{and} \quad I_n^+ := \{(j,k) : (j,k) \in I_n,\ j \le k\}. \tag{1.2}$$

d_n has the following natural interpretation: if the first n positive integers are evenly spread out on a circle of radius $\frac{n}{2\pi}$, then $d_n(j,k)$ is the distance between the integers j and k.

The quantity b_n will be the *radius* of a band of our random matrix, so that all entries of the matrix with $j,k \notin I_n$ are zero. Define a real symmetric band random matrix

$$M = (M_{jk}),\ 1 \le j,k \le n, \tag{1.3}$$

in such a way that for $j \le k$ one has

$$M_{jk} = M_{kj} = b_n^{-1/2} W_{jk} \ \text{ if } \ d_n(j,k) \le b_n, \tag{1.4}$$

and $M_{jk} = 0$ otherwise, where $\{W_{jk}\}_{(j,k)\in I_n^+}$ is a sequence of independent real valued random variables satisfying

$$\mathbb{E}\{W_{jk}\} = 0,\ \mathbb{E}\{W_{jk}^2\} = (1+\delta_{jk})\sigma^2. \tag{1.5}$$

In general, the distribution of the entries W_{jk} might depend on the size n of the matrix but we will not indicate this dependence it in our notations, unless it is necessary. An important special case corresponds to $b_n = \lfloor (n-1)/2 \rfloor$. Then M is standard Wigner random matrix (see e.g. [43], [4], [9], [1]).

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For a real symmetric (Hermitian) matrix M of order n, its empirical distribution of the eigenvalues is defined as $\mu_M = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$, where $\lambda_1 \leq \ldots \leq \lambda_n$ are the (ordered) eigenvalues of M. The Wigner semicircle law states that for any bounded continuous test function $\varphi : \mathbb{R} \to \mathbb{R}$, the linear statistic

$$\frac{1}{n}\sum_{i=1}^{n} \varphi(\lambda_i) = \frac{1}{n}\mathrm{Tr}(\varphi(M)) =: \mathrm{tr}_n(\varphi(M)) \tag{1.6}$$

converges to $\int \varphi(x) d\mu_{sc}(dx)$ in probability, where μ_{sc} is determined by its density

$$\frac{d\mu_{sc}}{dx}(x) = \frac{1}{4\pi\sigma^2}\sqrt{8\sigma^2 - x^2}\,\mathbf{1}_{[-2\sqrt{2}\sigma, 2\sqrt{2}\sigma]}(x). \tag{1.7}$$

We refer the reader to [43], [4], [9], [1] for the proof in the full matrix case and to [10], [28] for the proof in the band matrix case.

Band random matrices have important applications in physics (see e.g. [31], [14], [15], [21], [27], [40]), in particular as a model of quantum chaos. It is conjectured that the eigenvectors are localized and local eigenvalue statistics are Poisson for $b_n \ll \sqrt{n}$. On the other hand, it is expected that the eigenvectors are delocalized and local eigenvalue statistics follow GUE (GOE) law for $b_n \gg \sqrt{n}$ (see e.g. [21]). Throughout the paper, the relation $a_n \ll b_n$ for two n-dependent quantities a_n and b_n means that $a_n/b_n \to 0$ as $n \to \infty$. For recent mathematical progress on local spectral properties of band random matrices, we refer the reader to [18], [19], [20], [34], [38], [39].

The linear eigenvalues statistics corresponding to a test function φ is defined as

$$\mathscr{N}_n[\varphi] = \sum_{l=1}^{n} \varphi(\lambda_l). \tag{1.8}$$

In the Wigner (full matrix) case, the variance of $\mathscr{N}_n[\varphi]$ stays bounded as $n \to \infty$ for sufficiently smooth φ. Moreover, the fluctuation of the linear statistic is Gaussian in the limit (see e.g. [35], [2], [7], [25], [32], and references therein). Similar results have been established for other ensembles of random matrices ([22], [37], [33],[6]). In addition, we note recent results on partial linear eigenvalue statistics ([5], [29]).

In this paper, we prove that the normalized linear statistic

$$\mathscr{M}_n[\varphi] := (b_n/n)^{1/2}\mathscr{N}_n[\varphi] \tag{1.9}$$

has an asymptotic normal distribution, as $n \to \infty$ provided $b_n \ll \sqrt{n}$, and $\varphi,\ W_{jk}$ satisfy some conditions.

2 Statement of Main Results

First, we assume that the matrix entries satisfy the Poincaré inequality. We refer the reader to Section A of the Appendix for the definition and basic facts about the Poincaré inequality.

Theorem 2.1. *Let $M = W/\sqrt{b_n}$ be a real symmetric random band matrix (1.4-1.5), where $\{b_n\}$ is a sequence of integers satisfying $\sqrt{n} \ll b_n \ll n$. Assume the following:*

1. *Diagonal and non-zero off-diagonal entries of W are two sets of i.i.d random variables;*

2. *The marginal probability distribution of W_{jk} satisfies the Poincaré Inequality with some uniform constant $m > 0$ which does not depend on n, j, k;*

3. *The fourth moment of the non-zero off-diagonal entries does not depend on n:*

$$\mu_4 = \mathbb{E}\{W_{12}^4\}. \tag{2.1}$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a test function with continuous bounded derivative. Then the corresponding centered normalized linear statistic of the eigenvalues

$$\mathscr{M}_n^\circ[\varphi] := (b_n/n)^{1/2}\mathscr{N}_n^\circ[\varphi] = (b_n/n)^{1/2}(\mathscr{N}_n[\varphi] - \mathbb{E}\{\mathscr{N}_n[\varphi]\}) \tag{2.2}$$

converges in distribution to the Gaussian random variable with zero mean and the variance

$$\begin{aligned} Var_{band}[\varphi] &= \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma}\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma}\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{(\varphi(x)-\varphi(\lambda))\varphi'(y)\sqrt{8\sigma^2-x^2}\sqrt{8\sigma^2-y^2}}{4\pi^4(x-\lambda)\sqrt{8\sigma^2-\lambda^2}} F_\sigma(x,y)1_{\{x\neq y\}}dxdyd\lambda \\ &+ \frac{\kappa_4}{16\pi^2\sigma^8}\left(\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{\varphi(\lambda)(4\sigma^2-\lambda^2)}{\sqrt{8\sigma^2-\lambda^2}}d\lambda\right)^2, \end{aligned} \tag{2.3}$$

where for $x \neq y$

$$F_\sigma(x,y) := \int_{-\infty}^{\infty} \frac{\left(s^3\sin s - s\sin^3 s\right)ds}{2\sigma^2\left(s^2-\sin^2 s\right)^2 - \left(s^3\sin s + s\sin^3 s\right)xy + s^2\sin^2 s(x^2+y^2)}, \tag{2.4}$$

and κ_4 is the fourth cumulant of off-diagonal entries, i.e.

$$\kappa_4 = \mu_4 - 3\sigma^4. \tag{2.5}$$

Next, we extend this result to the non-i.i.d. case when the fifth moment of the matrix entries is uniformly bounded. Here we do not assume that marginal distributions of the non-zero entries satisfy the Poincaré inequality. For technical reasons, we assume that the fourth cumulant of the matrix entries is zero. Also we need $\sqrt{n}\ln n \ll b_n$ (thus, we have additional $\ln n$ factor at the l.h.s. as compared to the corresponding assumption in Theorem 2.1).

Theorem 2.2. *Let $M = W/\sqrt{b_n}$ be a real symmetric band matrix (1.4-1.5), where $\{b_n\}$ is a sequence of positive integers satisfying $\sqrt{n}\ln n \ll b_n \ll n$. Assume the following:*

1. $$\sigma_5 := \sup_{n\in\mathbb{N}}\max_{(j,k)\in I_n} \mathbb{E}\{|W_{jk}^{(n)}|^5\} < \infty. \tag{2.6}$$

2. *The third cumulant of the non-zero off-diagonal entries does not depend on j,k:*

$$\kappa_3 = \kappa_{3,jk}, \quad (j,k)\in I_n.$$

3. *The fourth cumulant of off-diagonal entries is zero: $\kappa_4 = \mathbb{E}\{(W_{jk}^{(n)})^4\} - 3\sigma^4 = 0$.*

Let $\varphi : \mathbb{R}\to\mathbb{R}$ be a test function with the Fourier transform

$$\hat\varphi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\lambda}\varphi(\lambda)d\lambda \tag{2.7}$$

satisfying

$$\int_{-\infty}^{\infty}(1+|t|^4)|\hat\varphi(t)|dt < \infty. \tag{2.8}$$

Then the corresponding centered normalized linear eigenvalues statistic $\mathscr{M}_n^\circ[\varphi]$ converges in distribution to the Gaussian random variable with zero mean and variance Var_G

$$Var_G[\varphi] = \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma}\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma}\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{(\varphi(x)-\varphi(\lambda))\varphi'(y)\sqrt{8\sigma^2-x^2}\sqrt{8\sigma^2-y^2}}{4\pi^4(x-\lambda)\sqrt{8\sigma^2-\lambda^2}} F_\sigma(x,y)1_{\{x\neq y\}}dxdyd\lambda. \tag{2.9}$$

Remark 2.3. *Similar results with little modification hold for Hermitian band random matrices. In particular, the variance (2.9) in Theorem 2.2 gets an additional factor* $1/2$, *provided (1.5) is replaced by*

$$\mathbb{E}\{W_{jk}\}=0,\ \mathbb{E}\{|W_{jk}|^2\}=(1+\delta_{jk})\sigma^2,\ \ \mathbb{E}\{W_{jk}^2\}=0. \tag{2.10}$$

The proofs are very similar and left to the reader.

The rest of the paper is organized as follows. We prove Theorem 2.1 in Section 3 and Theorem 2.2 in Section 4. In the Appendix, we list basic facts on the Poincaré inequality and decoupling formula.

3 Proof of Theorem 2.1

3.1 Stein's Method

We follow the approach used by A. Lytova and L. Pastur in [25] in the full matrix (Wigner) case. Essentially, it is a modification of the Stein's method ([41], [8]). While several steps of our proof are similar to the ones in [25], the fact that we are dealing with band matrices raises new significant difficulties (see e.g. Lemmas 3.11 and 3.12 in Subsection 3.5).

First, we prove the result of Theorem 2.1 under an additional technical condition on the smoothness of a test function. Namely, we assume that the Fourier transform of $\varphi:\mathbb{R}\to\mathbb{R}$ satisfies

$$\int_{-\infty}^{\infty}(1+|t|^{4+\varepsilon})|\hat{\varphi}(t)|dt<\infty, \tag{3.1}$$

where ε is an arbitrary small positive number. Once the result is established for such test functions, it can be easily extended to the case of functions with bounded continuous derivative using (3.10).

Let $Z_n(x), Z(x)$ be the characteristic functions of the normalized linear statistic (1.9) and the Gaussian distribution with zero mean and $Var_{band}[\varphi]$ variance, respectively, i.e.

$$Z_n(x)=\mathbb{E}\{e^{ix\mathscr{M}_n^\circ[\varphi]}\}, \tag{3.2}$$

and

$$Z(x)=\exp\{-x^2Var_{band}[\varphi]/2\}. \tag{3.3}$$

It is sufficient to show that for any $x\in\mathbb{R}$

$$\lim_{n\to\infty}Z_n(x)=Z(x). \tag{3.4}$$

We note that $Z(x)$ is the unique solution of the integral equation

$$Z(x)=1-Var_{band}[\varphi]\int_0^x yZ(y)dy \tag{3.5}$$

in the class of bounded continuous functions. It follows from (3.2) that the derivative of $Z_n(x)$ can be written as

$$Z_n'(x)=i\mathbb{E}\{\mathscr{M}_n^\circ[\varphi]e^{ix\mathscr{M}_n^\circ[\varphi]}\}. \tag{3.6}$$

To bound the derivative of Z_n, we use the Poincaré inequality. Since the Poincaré Inequality tensorises (see e.g. [1]), the joint distribution of $\{W_{jk}\}_{(j,k)\in I_n^+}$ on $\mathbb{R}^{n(b_n+1)}$ satisfies the Poincaré Inequality with the same constant $m>0$, i.e. for all continuously differentiable function Φ, we have

$$Var\{\Phi(\{W_{jk}\}_{(j,k)\in I_n^+})\}\le\frac{1}{m}\sum_{(j,k)\in I_n^+}\mathbb{E}\{|\frac{\partial\Phi}{\partial W_{jk}}(\{W_{jk}\})|^2\}. \tag{3.7}$$

Let

$$\beta_{jk}=(1+\delta_{jk})^{-1}=\begin{cases}1 & j\neq k,\\ 1/2 & j=k.\end{cases} \tag{3.8}$$

Since

$$\frac{\partial \mathscr{M}_n[\varphi]}{\partial W_{jk}} = \frac{2\beta_{jk}}{\sqrt{n}}\varphi'_{jk}(M), \tag{3.9}$$

we have

$$\begin{aligned} Var\{\mathscr{M}_n[\varphi]\} &\leq \frac{2}{mn}\mathbb{E}\{Tr(\varphi'(M)\varphi'(M)^*)\} \\ &\leq \frac{2}{m}(\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|)^2. \end{aligned} \tag{3.10}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$|Z_n'(x)| \leq \sqrt{\frac{2}{m}}\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|. \tag{3.11}$$

In addition, (3.10) implies

$$|Z_n''(x)| \leq \frac{2}{m}(\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|)^2. \tag{3.12}$$

Taking into account $Z_n(0) = 1$, we have

$$Z_n(x) = 1 + \int_0^x Z_n'(y)dy. \tag{3.13}$$

For any $T > 0$, the sequence $\{(Z_n(x), Z_n'(x))\}$ is pre-compact in $C([-T, T], \mathbb{R}^2)$. Therefore, it is enough to show that for any converging subsequence one has

$$\lim_{n_j\to\infty} Z'_{n_j}(x) = -xVar_{band}[\varphi]\lim_{n_j\to\infty} Z_{n_j}(x). \tag{3.14}$$

For the convenience of the reader, we use the same notations as in [25]:

$$D_{jk} := \partial/\partial M_{jk}; \tag{3.15}$$

$$U(t) := e^{itM}, U_{jk}(t) := (U(t))_{jk}; \tag{3.16}$$

$$u_n(t) := TrU(t),\ u_n^\circ(t) := u_n(t) - E\{u_n(t)\}. \tag{3.17}$$

Since $U(t)$ is a unitary matrix, we have

$$\|U\| = 1;\ |U_{jk}| \leq 1;\ \sum_{k=1}^n |U_{jk}|^2 = 1. \tag{3.18}$$

Moreover,

$$D_{jk}U_{ab}(t) = i\beta_{jk}(U_{aj} * U_{bk} + U_{ak} * U_{bj})(t), \tag{3.19}$$

where

$$f * g(t) := \int_0^t f(s)g(t-s)ds. \tag{3.20}$$

Applying the Fourier inversion formula

$$\varphi(\lambda) = \int_{-\infty}^{\infty} e^{it\lambda}\hat{\varphi}(t)dt, \tag{3.21}$$

we can write

$$\mathscr{M}_n^\circ[\varphi] = (b_n/n)^{1/2}\int_{-\infty}^{\infty}\hat{\varphi}(t)u_n^\circ(t)dt. \tag{3.22}$$

Therefore,

$$Z_n'(x) = i\int_{-\infty}^{\infty} \hat{\varphi}(t) Y_n(x,t)dt, \tag{3.23}$$

where

$$Y_n(x,t) := \mathbb{E}\{(b_n/n)^{1/2} u_n^\circ(t) e_n(x)\}, \tag{3.24}$$

and

$$e_n(x) = e^{ix\mathscr{M}_n^\circ[\varphi]}. \tag{3.25}$$

Taking into account (3.14) and (3.23), we conclude that the result of the theorem follows if we can establish the following two facts. First, we have to show that the sequence $\{Y_n\}$ is bounded and equicontinuous on any bounded subset of $\{t \geq 0, x \in \mathbb{R}\}$. Second, we have to show that any uniformly converging subsequence of Y_n has the same limit

$$Y(x,t) = \overline{Y(-x,-t)} \tag{3.26}$$

such that

$$i\int_{-\infty}^{\infty} \hat{\varphi}(t) Y(x,t)dt = -xVar_{band}[\varphi]Z(x). \tag{3.27}$$

The main technical part of the proof of Theorem 2.1 is the following proposition.

Proposition 3.1. *$Y_n(x,t)$ satisfies the equation*

$$\begin{aligned} Y_n(x,t) &+ \frac{2(2b_n+1)\sigma^2}{b_n}\int_0^t\int_0^{t_1} \bar{v}_n(t_1-t_2)Y_n(x,t_2)dt_2dt_1 \\ &= xZ_n(x)A_n(t) + 2i\kappa_4 xZ_n(x)\int_0^t \bar{v}_n * \bar{v}_n(t_1)dt_1 \int_{-\infty}^{\infty} t_2\bar{v}_n * \bar{v}_n(t_2)\hat{\varphi}(t_2)dt_2 + r_n(x,t), \end{aligned} \tag{3.28}$$

where

$$A_n(t) := -\frac{2\sigma^2}{n}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{U_{jk}(t_1)\varphi_{jk}'(M)\}dt_1, \tag{3.29}$$

$$\bar{v}_n(t) := n^{-1}\mathbb{E}Tre^{itM}, \tag{3.30}$$

$U_{jk}(t)$ is defined in (3.16), and $r_n(x,t)$ converges to zero uniformly on any bounded subset of $\{t \geq 0, x \in \mathbb{R}\}$.

The proof of Proposition 3.1 will be given in the remaining part of this subsection and in the next three subsections.

Proof. First, we show that $Y_n(x,t)$ is bounded and uniformly equicontinuous on bounded subsets of $\mathbb{R}^2$. Indeed, applying inequality (3.10) to $\varphi(\lambda) = e^{it\lambda}$ and $\varphi(\lambda) = i\lambda e^{it\lambda}$, we get

$$Var\{(b_n/n)^{1/2}u_n(t)\} \leq \frac{2t^2}{m} \tag{3.31}$$

and

$$Var\{(b_n/n)^{1/2}u_n'(t)\} \leq \frac{2}{m}(1+3\sigma^2t^2). \tag{3.32}$$

This implies

$$|Y_n(x,t)| \leq Var^{1/2}\{(b_n/n)^{1/2}u_n(t)\} \leq \sqrt{\frac{2}{m}}|t|, \tag{3.33}$$

$$|\frac{\partial}{\partial t}Y_n(x,t)| \leq Var^{1/2}\{(b_n/n)^{1/2}u_n'(t)\} \leq \sqrt{\frac{2}{m}(1+3\sigma^2t^2)}, \tag{3.34}$$

and

$$|\frac{\partial}{\partial x}Y_n(x,t)| \le \frac{2}{m}|t|\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|. \tag{3.35}$$

Therefore, we have shown that $\{Y_n\}$ is bounded and equicontinuous on any bounded subset of $\mathbb{R}^2$. Applying the identity $e^{itM} = 1 + i\int_0^t Me^{isM}ds$, we have

$$u_n(t) = n + i\int_0^t \sum_{(j,k)\in I_n} M_{jk}U_{jk}(t_1)dt_1, \tag{3.36}$$

and

$$Y_n(x,t) = \frac{i}{\sqrt{n}}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{W_{jk}U_{jk}(t_1)e_n^\circ(x)\}dt_1, \tag{3.37}$$

where $e_n^\circ = e_n - \mathbb{E}\{e_n\}$. To analyze (3.37), we use the decoupling formula (B.1) with $p = 3$ to obtain

$$Y_n(x,t) = \frac{i}{\sqrt{n}}\int_0^t \sum_{(j,k)\in I_n} \left\{\sum_{l=0}^{3}\frac{\kappa_{l+1,jk}}{b_n^{l/2}l!}\mathbb{E}\{D_{jk}^l(U_{jk}(t_1)e_n^\circ(x))\} + \varepsilon_{3,jk}\right\}dt_1, \tag{3.38}$$

where $\kappa_{l,jk}$ is the lth cumulant of W_{jk}, i.e.

$$\kappa_{1,jk} = 0,\ \kappa_{2,jk} = (1+\delta_{jk})\sigma^2;$$

for $j \ne k$

$$\kappa_{3,jk} = \mathbb{E}\{(W_{12}^{(n)})^3\} =: \kappa_3,\ \kappa_{4,jk} = \kappa_4;$$

for $j = k$

$$\kappa_{3,jj} = \mathbb{E}\{(W_{11}^{(n)})^3\} =: \kappa_3',\ \kappa_{4,jj} = \mathbb{E}\{(W_{11}^{(n)})^4\} - 12\sigma^2 =: \kappa_4'.$$

In addition, we note that the remainder term $\varepsilon_{3,jk}$ in (3.38) is bounded as

$$|\varepsilon_{3,jk}| \le C_3\mathbb{E}\{|W_{jk}|^5\}\sup_{W_{jk}\in\mathbb{R}}\left|\frac{D_{jk}^4U_{jk}(t)e_n^\circ(x)}{b_n^2}\right|. \tag{3.39}$$

Since the marginal distribution of the matrix entries satisfies the PI with constant m independent of n, the third and fourth cumulants are uniformly bounded in n, i.e. there exist σ_3, σ_4 independent of n such that

$$|\kappa_3|, |\kappa_3'| \le \sigma_3, |\kappa_4'| \le \sigma_4. \tag{3.40}$$

and $\sigma_5 := \max_{j,k,n}\mathbb{E}\{|W_{jk}|^5\} < \infty$.

We need the following technical lemma.

Lemma 3.2.

$$|D_{jk}^l(U_{jk}(t)e_n^\circ(x))| \le C_l(\sqrt{b_n/n}x, t),\ l = 1,2,3,4, \tag{3.41}$$

where $C_l(x,t)$ is some polynomial in $|x|, |t|$ of degree l with positive coefficients independent of n.

Proof. (3.19) implies

$$|D_{jk}^lU_{jk}(t)| \le c_l|t|^l. \tag{3.42}$$

In addition,

$$D_{jk}e_n(x) = -2(b_n/n)^{1/2}\beta_{jk}xe_n(x)\int_{-\infty}^{\infty}sU_{jk}(s)\hat{\varphi}(s)ds = 2i(b_n/n)^{1/2}\beta_{jk}xe_n(x)\varphi_{jk}'(M). \tag{3.43}$$

It follows from (3.1) that φ has fourth bounded derivative. Thus, for $l = 1,2,3,4$

$$|D_{jk}^le_n(x)| \le c_l'(1 + |(b_n/n)^{1/2}x|^l). \tag{3.44}$$

Combining (3.42) and (3.44) we obtain Lemma 3.2. □

Lemma 3.2 and (3.39) imply

$$|\varepsilon_{3jk}| \le C_3\sigma_5 C_4((b_n/n)^{1/2}x,t)/b_n^2. \tag{3.45}$$

We can rewrite (3.38) as

$$Y_n(x,t) = T_1 + T_2 + T_3 + \mathscr{E}_3, \tag{3.46}$$

where

$$T_l := \frac{i}{l!\sqrt{nb_n^l}} \int_0^t \sum_{(j,k)\in I_n} \kappa_{l+1,jk}\mathbb{E}\{D_{jk}^l(U_{jk}(t_1)e_n^\circ(x))\}dt_1,\ \ l=1,2,3, \tag{3.47}$$

and

$$\mathscr{E}_3 = in^{-1/2}\int_0^t \sum_{(j,k)\in I_n} \varepsilon_{3,jk}dt_1. \tag{3.48}$$

By (3.45), we have

$$|\mathscr{E}_3| \le \frac{\sqrt{n}}{b_n} C_5((b_n/n)^{1/2}x,t). \tag{3.49}$$

Since $n/b_n^2 \to 0$, we obtain that $\mathscr{E}_3 \to 0$ on any bounded subset of $\mathbb{R}^2$ as $n\to\infty$. In the next three subsections, we consider separately each of the terms $T_l,\ \ l=1,2,3$ in (3.46) and finish the proof of Proposition 3.1.

3.2 Estimate of T_1

The main result of this subsection is contained in the following proposition.

Proposition 3.3. *Let T_1 be defined as in (3.47) with $l=1$. Then*

$$T_1 = -\frac{2(2b_n+1)\sigma^2}{b_n}\int_0^t\int_0^{t_1} \bar{v}_n(t_1-t_2)Y_n(x,t_2)dt_2dt_1 + xZ_n(x)A_n(t) + \varepsilon_n(x,t), \tag{3.50}$$

where $\bar{v}_n(t)$ is defined in (3.30), $A_n(t)$ is defined in (3.29), and $\varepsilon_n(x,t)\to 0$ as $n\to\infty$ uniformly on any bounded subset of $\{(x,t), t\ge 0\}$.

Proof. First, by (3.19) we write

$$T_1 = T_{11} + T_{12} + T_{13}, \tag{3.51}$$

where

$$T_{11} = -\frac{\sigma^2}{\sqrt{nb_n}}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{U_{jk} * U_{jk}(t_1)e_n^\circ(x)\}dt_1, \tag{3.52}$$

$$T_{12} = -\frac{\sigma^2}{\sqrt{nb_n}}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{U_{jj} * U_{kk}(t_1)e_n^\circ(x)\}dt_1, \tag{3.53}$$

$$T_{13} = -\frac{2\sigma^2}{n}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{U_{jk}(t_1)xe_n(x)\varphi_{jk}'(M)\}dt_1. \tag{3.54}$$

It follows from the Cauchy-Schwarz inequality and $|e_n(x)|\le 1$ that

$$|T_{11}| \le \frac{\sigma^2}{\sqrt{nb_n}}\int_0^t \sum_{k=1}^n Var^{1/2}\left\{\sum_{j:(j,k)\in I_n} (U_{jk} * U_{jk})(t_1)\right\}dt_1. \tag{3.55}$$

Let us fix k and define

$$U^{(k)}(t) := (U_{jl}^{(k)}(t))_{j,l=1,\dots,n}, \tag{3.56}$$

where

$$U_{jl}^{(k)}(t) = \begin{cases} U_{jl}(t) & \text{if } (j,k) \in I_n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.57}$$

Then

$$\|U^{(k)}(t)\| \leq 1 \tag{3.58}$$

By the Poincaré Inequality (3.7), (3.8), (3.19), and the Cauchy-Schwarz Inequality, we have

$$Var\{\sum_{j:(j,k)\in I_n} U_{jk} * U_{jk}(t_1)\} \leq \frac{4}{mb_n} \sum_{(p,s)\in I_n^+} \mathbb{E}\{|\sum_{j:(j,k)\in I_n} U_{jp} * U_{ks} * U_{jk}(t_1) + U_{js} * U_{kp} * U_{jk}(t_1)|^2\}$$

$$\leq \frac{8t_1^2}{mb_n}\mathbb{E}\{\int_0^{t_1}\int_0^{t_2}\sum_{p=1}^n |(U(t_1-t_2)U^{(k)}(t_3))_{pk}|^2 \sum_{s=1}^n |U_{ks}(t_2-t_3)|^2 dt_3 dt_2\}. \tag{3.59}$$

It follows from (3.18) and

$$\sum_{p=1}^n \left|(U(t_1-t_2)U^{(k)}(t_3))_{pk}\right|^2 = \|U(t_1-t_2)U^{(k)}(t_3)e_k\|^2 = \|U^{(k)}(t_3)e_k\|^2 \leq \|e_k\|^2 = 1 \tag{3.60}$$

that we have

$$Var\{\sum_{j:(j,k)\in I_n} U_{jk} * U_{jk}(t_1)\} \leq \frac{4t_1^4}{mb_n}. \tag{3.61}$$

Hence,

$$|T_{11}| \leq \frac{2\sqrt{n}\sigma^2 t^3}{3\sqrt{m}b_n}. \tag{3.62}$$

Recall that $n/b_n^2 \to 0$, so $T_{11} \to 0$ as $n \to \infty$ if t is bounded.

Now, we turn out attention to (3.53). We write T_{12} as follows

$$T_{12} = -\frac{2\sigma^2}{\sqrt{nb_n}}\int_0^t\int_0^{t_1}\sum_{k=1}^n \left\{\mathbb{E}\{U_{kk}(t_1-t_2)\}\sum_{j:(j,k)\in I_n}\mathbb{E}\{U_{jj}(t_2)e_n^\circ(x)\}\right\} dt_2 dt_1 + T_{12}', \tag{3.63}$$

where

$$T_{12}' = -\frac{\sigma^2}{\sqrt{nb_n}}\int_0^t\int_0^{t_1}\sum_{(j,k)\in I_n}\mathbb{E}\{U_{kk}^\circ(t_1-t_2)U_{jj}^\circ(t_2)e_n^\circ(x)\}dt_2 dt_1. \tag{3.64}$$

Since $\mathbb{E}\{U_{kk}(t)\}$ and $\mathbb{E}\{U_{kk}(t)e_n^\circ(x)\}$ are k-independent, $\mathbb{E}\{U_{kk}(t)\} = \bar{v}_n(t)$, and

$$\sum_{j:(j,k)\in I_n}\mathbb{E}\{U_{jj}(t_2)e_n^\circ(x)\} = \frac{2b_n+1}{n}\mathbb{E}\{u(t_2)e_n^\circ(x)\} = \frac{2b_n+1}{\sqrt{nb_n}}Y_n(x,t). \tag{3.65}$$

Thus, the first term in (3.63) can be written as

$$-\frac{2(2b_n+1)\sigma^2}{b_n}\int_0^t\int_0^{t_1}\bar{v}_n(t_1-t_2)Y_n(x,t_2)dt_2 dt_1. \tag{3.66}$$

We are left to bound T_{12}'. The Cauchy-Schwarz inequality and $|e_n^\circ(x)| \leq 2$ imply

$$|T_{12}'| \leq \frac{2\sigma^2}{\sqrt{nb_n}}\int_0^t\int_0^{t_1}\sum_{k=1}^n Var^{1/2}\{U_{kk}(t_2)\}Var^{1/2}\{\sum_{j:(j,k)\in I_n}U_{jj}(t_1-t_2)\}dt_2 dt_1. \tag{3.67}$$

Applying (3.7), (3.19), (3.8) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} Var\{U_{jk}(t)\} &\le \frac{4}{mb_n}\sum_{(p,s)\in I_n^+}\mathbb{E}\{|U_{jp}*U_{ks}(t)|^2\} \le \frac{4t}{mb_n}\mathbb{E}\{\int_0^t \sum_{(p,s)\in I_n^+}|U_{jp}(t_1)U_{ks}(t-t_1)|^2dt_1\} \\ &\le \frac{4t}{mb_n}\mathbb{E}\{\int_0^t\sum_{p=1}^n|U_{jp}(t_1)|^2\sum_{s=1}^n|U_{ks}(t-t_1)|^2dt_1\} = \frac{4t^2}{mb_n}, \end{aligned} \tag{3.68}$$

and for fixed k,

$$\begin{aligned} Var\{\sum_{j:(j,k)\in I_n}U_{jj}(t)\} &\le \frac{4}{mb_n}\sum_{(p,s)\in I_n^+}\mathbb{E}\{|\sum_{j:(j,k)\in I_n}U_{jp}*U_{js}(t)|^2\} \\ &= \frac{4}{mb_n}\sum_{(p,s)\in I_n^+}\mathbb{E}\{|\int_0^t\sum_{j:(j,k)\in I_n}U_{jp}(t_1)U_{js}(t-t_1)dt_1|^2\} \\ &= \frac{4}{mb_n}\sum_{(p,s)\in I_n^+}\mathbb{E}\{|\int_0^t(U(t_1)U^{(k)}(t-t_1))_{ps}dt_1|^2\} \\ &\le \frac{4t}{mb_n}\mathbb{E}\{\int_0^t\sum_{p,s=1}^n|(U(t_1)U^{(k)}(t-t_1))_{ps}|^2dt_1\} \\ &= \frac{4t}{mb_n}\mathbb{E}\{\int_0^t TrU(t_1)U^{(k)}(t-t_1)U(t_1)^*U^{(k)}(t-t_1)^*dt_1\} \\ &= \frac{4t}{mb_n}\mathbb{E}\{\int_0^t TrU^{(k)}(t-t_1)U^{(k)}(t-t_1)^*dt_1\}. \end{aligned} \tag{3.69}$$

Since

$$Tr(U^{(k)}(t)U^{(k)}(t)^*) \le (2b_n+1)\|U^{(k)}(t)U^{(k)}(t)^*\| \le (2b_n+1)\|U^{(k)}(t)\|^2 \le 3b_n, \tag{3.70}$$

we conclude that

$$Var\left\{\sum_{j:(j,k)\in I_n}U_{jj}(t)\right\} \le \frac{12t^2}{m}. \tag{3.71}$$

Therefore,

$$|T'_{12}| \le \frac{2\sigma^2}{\sqrt{nb_n}}\int_0^t\int_0^{t_1} n\sqrt{\frac{4t_2^2}{mb_n}}\sqrt{\frac{12(t_1-t_2)^2}{m}}dt_2dt_1 = \frac{Const\sigma^2t^4\sqrt{n}}{mb_n}. \tag{3.72}$$

Now, we turn our attention to T_{13}. We can rewrite (3.54) in the following form

$$T_{13} = xZ_n(x)A_n(t) + T'_{13}, \tag{3.73}$$

where $Z_n(x)$ is given by (3.2), $A_n(t)$ is defined in (3.29), and

$$T'_{13} = -\frac{2i\sigma^2x}{n}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{U_{jk}(t_1)e_n^\circ(x)\int_{-\infty}^{\infty}t_2U_{jk}(t_2)\hat{\varphi}(t_2)dt_2\}dt_1. \tag{3.74}$$

Then

$$|T'_{13}| \le \frac{2\sigma^2x}{n}\int_0^t\int_{-\infty}^{\infty}\sum_{k=1}^n Var^{1/2}\left\{\sum_{j:(j,k)\in I_n}U_{jk}(t_1)U_{jk}(t_2)\right\}|t_2||\hat{\varphi}(t_2)|dt_2dt_1. \tag{3.75}$$

Let us fix k. The Poincaré Inequality (3.7), together with (3.19) and (3.8) imply

$$
\begin{aligned}
&Var\left\{\sum_{j:(j,k)\in I_n} U_{jk}(t_1)U_{jk}(t_2)\right\} \\
&\le \frac{1}{mb_n}\sum_{(p,s)\in I_n^+}\mathbb{E}\left\{\left|\sum_{j:(j,k)\in I_n}[U_{jp}*U_{ks}(t_1)+U_{js}*U_{kp}(t_1)]U_{jk}(t_2)+U_{jk}(t_1)[U_{jp}*U_{ks}(t_2)+U_{js}*U_{kp}(t_2)]\right|^2\right\} \\
&\le \frac{4}{mb_n}\sum_{(p,s)\in I_n^+}\mathbb{E}\left\{|\sum_{j:(j,k)\in I_n} U_{jp}*U_{ks}(t_1)U_{jk}(t_2)|^2+|\sum_{j:(j,k)\in I_n} U_{js}*U_{kp}(t_1)U_{jk}(t_2)|^2\right. \\
&\quad \left.+|\sum_{j:(j,k)\in I_n} U_{jk}(t_1)U_{jp}*U_{ks}(t_2)|^2+|\sum_{j:(j,k)\in I_n} U_{jk}(t_1)U_{js}*U_{kp}(t_2))|^2\right\} \\
&\le \frac{8}{mb_n}\sum_{p,s=1}^{n}\mathbb{E}\left\{|\sum_{j:(j,k)\in I_n} U_{jp}*U_{ks}(t_1)U_{jk}(t_2)|^2+|\sum_{j:(j,k)\in I_n} U_{jk}(t_1)U_{jp}*U_{ks}(t_2)|^2\right\}. \qquad (3.76)
\end{aligned}
$$

Note that the Cauchy-Schwarz inequality gives us

$$
\begin{aligned}
&|\sum_{j:(j,k)\in I_n} U_{jp}*U_{ks}(t_1)U_{jk}(t_2)|^2 = |\int_0^{t_1}\sum_{j:(j,k)\in I_n} U_{jp}(t_3)U_{ks}(t_1-t_3)U_{jk}(t_2)dt_3|^2 \\
&= |\int_0^{t_1}(U(t_3)U^{(k)}(t_2))_{pk}U_{ks}(t_1-t_3)dt_3|^2 \le t_1\int_0^{t_1}|(U(t_3)U^{(k)}(t_2))_{pk}U_{ks}(t_1-t_3)|^2dt_3. \qquad (3.77)
\end{aligned}
$$

Using (3.18) and (3.60), we obtain

$$
\sum_{p,s=1}^{n}|\sum_{j:(j,k)\in I_n} U_{jp}*U_{ks}(t_1)U_{jk}(t_2)|^2 \le t_1\int_0^{t_1}\sum_{p=1}^{n}|(U(t_3)U^{(k)}(t_2))_{pk}|^2\sum_{s=1}^{n}|U_{ks}(t_1-t_3)|^2dt_3 \le \frac{t_1^2}{2}. \qquad (3.78)
$$

Hence,

$$
Var\{\sum_{j:(j,k)\in I_n} U_{jk}(t_1)U_{jk}(t_2)\} \le \frac{4(t_1^2+t_2^2)}{mb_n}. \qquad (3.79)
$$

Therefore,

$$
\begin{aligned}
|T_{13}'| &\le 2\sigma^2|x|\int_0^t\int_{-\infty}^{\infty}|t_2|\sqrt{\frac{4(t_1^2+t_2^2)}{mb_n}}|\hat{\varphi}(t_2)|dt_2dt_1 \\
&= 2\sigma^2|x|\int_0^t\left[\int_{|t_2|\le t_1}|t_2|\sqrt{\frac{4(t_1^2+t_2^2)}{mb_n}}|\hat{\varphi}(t_2)|dt_2+\int_{|t_2|>t_1}|t_2|\sqrt{\frac{4(t_1^2+t_2^2)}{mb_n}}|\hat{\varphi}(t_2)|dt_2\right]dt_1 \\
&\le \frac{4\sqrt{2}\sigma^2|x|}{\sqrt{mb_n}}\int_0^t\left[t_1^2\int_{|t_2|\le t_1}|\hat{\varphi}(t_2)|dt_2+\int_{|t_2|>t_1}|t_2|^2|\hat{\varphi}(t_2)|dt_2\right]dt_1 \\
&\le \frac{\sigma^2|x|}{\sqrt{mb_n}}C_3(t). \qquad (3.80)
\end{aligned}
$$

Combining the bounds obtained in this subsection, we get

$$
T_1 = -\frac{2(2b_n+1)\sigma^2}{b_n}\int_0^t\int_0^{t_1}\bar{v}_n(t_1-t_2)Y_n(x,t_2)dt_2dt_1 + xZ_n(x)A_n(t)+\varepsilon_n(x,t). \qquad (3.81)
$$

Since $b_n/n \to 0$, $n/b_n^2 \to 0$, using (3.62), (3.72), and (3.80), we have that $\varepsilon_n(x,t) = T_{11} + T'_{12} + T'_{13} \to 0$ on any bounded subset of $\{(x,t), t \geq 0\}$. Proposition 3.3 is proven. $\square$

3.3 Estimate of T_2

The main result of this subsection is the following proposition.

Proposition 3.4. *Let T_2 be defined as in (3.47) with $l = 2$. Then T_2 converges to zero as $n \to \infty$ uniformly on any bounded subset of $\{t \geq 0, x \in \mathbb{R}\}$.*

Proof.

$$T_2 = \frac{i\kappa_3}{2\sqrt{n}b_n}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{D_{jk}^2(U_{jk}(t_1)e_n^\circ(x))\}dt_1 + \frac{i(\kappa_3' - \kappa_3)}{2\sqrt{n}b_n}\int_0^t \sum_{j=1}^n \mathbb{E}\{D_{jj}^2(U_{jj}(t_1)e_n^\circ(x))\}dt_1. \quad (3.82)$$

By Lemma 3.2, the second term in T_2 is bounded by $\frac{\sqrt{n}}{b_n}C_3((b_n/n)^{1/2}x,t)$. As for the first term in T_2, it can be written as the sum of T_{21} and T_{22}, where

$$\begin{aligned} T_{21} &= -\frac{i\kappa_3}{\sqrt{n}b_n}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{\beta_{jk}^2(U_{jk} * U_{jk} * U_{jk})(t_1)e_n^\circ(x)\}dt_1 \\ &+\frac{2\kappa_3}{n\sqrt{b_n}}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{\beta_{jk}^2(U_{jk} * U_{jk})(t_1)xe_n(x)\int_{-\infty}^{\infty} t_2U_{jk}(t_2)\hat{\varphi}(t_2)dt_2\}dt_1 \\ &+\frac{2i\kappa_3}{n^{3/2}}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{\beta_{jk}^2x^2U_{jk}(t_1)e_n(x)(\int_{-\infty}^{\infty} t_2U_{jk}(t_2)\hat{\varphi}(t_2)dt_2)^2\}dt_1 \\ &+\frac{\kappa_3}{n\sqrt{b_n}}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{\beta_{jk}^2xU_{jk}(t_1)e_n(x)\int_{-\infty}^{\infty} t_2(U_{jk} * U_{jk})(t_2)\hat{\varphi}(t_2)dt_2\}dt_1, \end{aligned} \quad (3.83)$$

$$\begin{aligned} T_{22} &= -\frac{i\kappa_3}{\sqrt{n}b_n}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{\beta_{jk}^2(3U_{jj} * U_{jk} * U_{kk})(t_1)e_n^\circ(x)\}dt_1 \\ &+\frac{2\kappa_3}{n\sqrt{b_n}}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{\beta_{jk}^2(U_{jj} * U_{kk})(t_1)xe_n(x)\int_{-\infty}^{\infty} t_2U_{jk}(t_2)\hat{\varphi}(t_2)dt_2\}dt_1 \\ &+\frac{\kappa_3}{n\sqrt{b_n}}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{\beta_{jk}^2xU_{jk}(t_1)e_n(x)\int_{-\infty}^{\infty} t_2(U_{jj} * U_{kk})(t_2)\hat{\varphi}(t_2)dt_2\}dt_1. \end{aligned} \quad (3.84)$$

Since for any $s_1, s_2, s_3 \in \mathbb{R}$, one has

$$|\sum_{(j,k)\in I_n} U_{jk}(s_1)U_{jk}(s_2)U_{jk}(s_3)| \leq \sum_{j,k=1}^n |U_{jk}(s_1)U_{jk}(s_2)| \leq \left(\sum_{j,k=1}^n |U_{jk}(s_1)|^2\right)^{1/2}\left(\sum_{j,k=1}^n |U_{jk}(s_2)|^2\right)^{1/2} = n. \quad (3.85)$$

we have

$$|T_{21}| \leq \sqrt{n}/b_nC_2(\sqrt{b_n/n}x)t. \quad (3.86)$$

To estimate T_{22}, we note that

$$\left|\mathbb{E}\left\{\sum_{(j,k)\in I_n} \beta_{jk}^2U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)e_n^\circ(x)\right\}\right| \leq Var^{1/2}\left\{\sum_{(j,k)\in I_n} \beta_{jk}^2U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)\right\}. \quad (3.87)$$

Using the Poincaré inequality, we obtain an upper bound

$$\begin{aligned}
&Var\left\{\sum_{(j,k)\in I_n}\beta_{jk}^2 U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)\right\}\\
&\le \frac{12}{mb_n}\sum_{(p,q)\in I_n^+}\mathbb{E}\left\{|\sum_{(j,k)\in I_n}U_{jp}*U_{jq}(s_1)U_{jk}(s_2)U_{kk}(s_3)|^2+|\sum_{(j,k)\in I_n}U_{jj}(s_1)U_{jp}*U_{kq}(s_2)U_{kk}(s_3)|^2\right.\\
&\left.+|\sum_{(j,k)\in I_n}U_{jj}(s_1)U_{jq}*U_{kp}(s_2)U_{kk}(s_3)|^2+|\sum_{(j,k)\in I_n}U_{jj}(s_1)U_{jk}(s_2)U_{kp}*U_{kq}(s_3)|^2\right\}. \qquad (3.88)
\end{aligned}$$

Let $\alpha_j=\sum_{k:(j,k)\in I_n}U_{jk}(s_2)U_{kk}(s_3)$ and $D=diag\{\alpha_1,...,\alpha_n\}$. Then $|\alpha_j|\le\sqrt{2b_n+1}$, and

$$\begin{aligned}
&\sum_{(p,q)\in I_n^+}|\sum_{(j,k)\in I_n}U_{jp}*U_{jq}(s_1)U_{jk}(s_2)U_{kk}(s_3)|^2\le\sum_{(p,q)\in I_n^+}|s_1|\int_0^{s_1}|\sum_{(j,k)\in I_n}U_{jp}(t)U_{jq}(s_1-t)U_{jk}(s_2)U_{kk}(s_3)|^2dt\\
&=|s_1|\int_0^{s_1}\sum_{(p,q)\in I_n^+}|\sum_{j=1}^n U_{jp}(t)U_{jq}(s_1-t)\alpha_j|^2dt=|s_1|\int_0^{s_1}\sum_{(p,q)\in I_n^+}|(U(t)DU(s_1-t))_{pq}|^2dt\\
&\le|s_1|\int_0^{s_1}n\|U(t)DU(s_1-t)\|^2dt\le s_1^2n(2b_n+1). \qquad (3.89)
\end{aligned}$$

Now let $D_1=diag\{U_{11}(s_1),...,U_{nn}(s_1)\}$, $D_2=diag\{U_{11}(s_3),...,U_{nn}(s_3)\}$, and $B=(B_{j,k})_{j,k=1}^n$ be a 0-1 band matrix, such that $B_{jk}=1_{(j,k)\in I_n}$. Then

$$\begin{aligned}
&\sum_{(p,q)\in I_n^+}|\sum_{(j,k)\in I_n}U_{jj}(s_1)U_{jp}*U_{kq}(s_2)U_{kk}(s_3)|^2\le\sum_{(p,q)\in I_n^+}|s_2|\int_0^{s_2}|\sum_{(j,k)\in I_n}U_{jj}(s_1)U_{jp}(t)U_{kq}(s_2-t)U_{kk}(s_3)|^2dt\\
&=|s_2|\int_0^{s_2}\sum_{(p,q)\in I_n^+}|(U(t)D_1BD_2U(s_2-t))_{pq}|^2dt\le s_2^2n\|B\|^2=s_2^2n(2b_n+1)^2. \qquad (3.90)
\end{aligned}$$

Hence,

$$Var\{\sum_{(j,k)\in I_n}\beta_{jk}^2U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)\}\le\frac{12n(2b_n+1)}{mb_n}(s_1^2+s_3^2+2s_2^2(2b_n+1)) \qquad (3.91)$$

$$\le nb_nC_2(s_1,s_2,s_3), \qquad (3.92)$$

$$|\mathbb{E}\{\sum_{(j,k)\in I_n}\beta_{jk}^2U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)e_n^\circ(x)\}|\le\sqrt{nb_n}C_2^{1/2}(s_1,s_2,s_3). \qquad (3.93)$$

For the last two terms in T_{22}, taking into account that

$$\mathbb{E}\{U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)e_n(x)\}=\mathbb{E}\{U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)e_n^\circ(x)\}+\mathbb{E}\{U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)\}\mathbb{E}\{e_n(x)\}, \qquad (3.94)$$

we have

$$|\sum_{(j,k)\in I_n}\beta_{jk}^2\mathbb{E}\{U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)e_n(x)\}|\le\sqrt{nb_n}C_2^{1/2}(s_1,s_2,s_3)+n/4+\sum_{(j,k)\in I_n,j\ne k}|\mathbb{E}\{U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)\}|. \qquad (3.95)$$

For $j\ne k$, we can write

$$\begin{aligned}
&\mathbb{E}\{U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)\}=\mathbb{E}\{U_{jj}^\circ(s_1)U_{jk}(s_2)U_{kk}^\circ(s_3)\}+\mathbb{E}\{U_{jj}(s_1)\}\mathbb{E}\{U_{jk}^\circ(s_2)U_{kk}^\circ(s_3)\}\\
&+\mathbb{E}\{U_{jj}^\circ(s_1)U_{jk}^\circ(s_2)\}\mathbb{E}\{U_{kk}(s_3)\}+\mathbb{E}\{U_{jj}(s_1)\}\mathbb{E}\{U_{jk}(s_2)\}\mathbb{E}\{U_{kk}(s_3)\}. \qquad (3.96)
\end{aligned}$$

So the left hand side of (3.96) is bounded by

$$(Var\{U_{jj}(s_1)\}Var\{U_{kk}(s_2)\})^{1/2}+(Var\{U_{kk}(s_3)\}Var\{U_{jk}(s_2)\})^{1/2}+(Var\{U_{jj}(s_1)\}Var\{U_{jk}(s_2)\})^{1/2}+|\mathbb{E}\{U_{jk}(s_2)\}|. \tag{3.97}$$

By (3.68) it is bounded from above by

$$\frac{4(|s_1s_2|+|s_1s_3|+|s_2s_3|)}{mb_n}+|\mathbb{E}\{U_{jk}(s_2)\}|. \tag{3.98}$$

To bound the second term in the last expression, we use the following auxiliary proposition.

Proposition 3.5. *Let $M = W/\sqrt{b_n}$ be a real symmetric band random matrix defined as Theorem 2.1 and $U(t) = e^{itM}$. Then*

$$\sup_{j\neq k}|\mathbb{E}\{U_{jk}(t)\}| = O\left(\frac{1+t^6}{b_n}\right). \tag{3.99}$$

The proof of Proposition 3.5 is given in Appendix D. Assuming (3.99), we conclude that

$$|\mathbb{E}\{U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)\}| \leq \frac{4(|s_1s_2|+|s_1s_3|+|s_2s_3c|)}{mb_n}+O(\frac{1+s_2^6}{b_n}). \tag{3.100}$$

Therefore,

$$|\sum_{(j,k)\in I_n}\beta_{jk}^2\mathbb{E}\{U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)e_n(x)\}| \leq \sqrt{nb_n}C_2^{1/2}(s_1,s_2,s_3)+n/4+\frac{4n(|s_1s_2b|+|s_1s_3c|+|s_2s_3c|)}{m}+O(s_2^6n), \tag{3.101}$$

and

$$|T_{22}| \leq \frac{\sqrt{n}}{b_n}C_4(t)+\frac{|x|}{b_n^{\varepsilon/6}}C_4(t). \tag{3.102}$$

This and (3.86) imply that T_2 converges to zero uniformly on any bounded subset of $\{t\geq 0, x\in\mathbb{R}\}$. Proposition 3.4 is proven. □

3.4 Estimate of T_3.

Finally, let us consider T_3. The main result of this subsection is the following bound.

Proposition 3.6. *Let T_3 be defined as in (3.47) (with $l=3$). Then*

$$T_3(x,t) = 2i\kappa_4 xZ_n(x)\int_0^t \bar{v}_n * \bar{v}_n(t_1)dt_1\int_{-\infty}^{\infty}t_2\bar{v}_n*\bar{v}_n(t_2)\hat{\varphi}(t_2)dt_2+\epsilon_n(x,t), \tag{3.103}$$

where $\epsilon_n(x,t)\to 0$ as $n\to\infty$ uniformly on any bounded subset of $\{(x,t), t\geq 0\}$.

Proof. One has

$$T_3(x,t) = \frac{i\kappa_4}{6\sqrt{nb_n^3}}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{D_{jk}^3(U_{jk}(t_1)e_n^\circ(x))\}dt_1+T_3'(x,t), \tag{3.104}$$

where the term T_3' comes from the summation over $j=k$ and corresponds to the fact that the marginal distribution of the diagonal entries is different from the marginal distribution of the off-diagonal entries. It follows form Lemma 3.2 that T_3' can be bounded as

$$|T_3'(x,t)| \leq \sqrt{\frac{n}{b_n^3}}C_4((b_n/n)^{1/2}x,t). \tag{3.105}$$

By (3.19), the first term in (3.104) can be written as the sum of

$$T_{31}(x,t)=\frac{\kappa_4}{\sqrt{nb_n^3}}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{\beta_{jk}^3(U_{jj}*U_{jj}*U_{kk}*U_{kk})(t_1)e_n^\circ(x)\}dt_1 \tag{3.106}$$

$$+\frac{i\kappa_4}{nb_n}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{\beta_{jk}^3(U_{jj}*U_{kk})(t_1)xe_n(x)\int_{-\infty}^{\infty}t_2(U_{jj}*U_{kk})(t_2)\hat{\varphi}(t_2)dt_2\}dt_1, \tag{3.107}$$

and

$$\begin{aligned}
T_{32}(x,t)&=\frac{\kappa_4}{\sqrt{nb_n^3}}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{\beta_{jk}^3(6U_{jj}*U_{jk}*U_{jk}*U_{kk}+U_{jk}*U_{jk}*U_{jk}*U_{jk})(t_1)e_n^\circ(x)\}dt_1\\
&+\frac{i\kappa_4}{nb_n}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{\beta_{jk}^3(6U_{jj}*U_{jk}*U_{kk}+2U_{jk}*U_{jk}*U_{jk})(t_1)xe_n(x)\int_{-\infty}^{\infty}t_2U_{jk}(t_2)\hat{\varphi}(t_2)dt_2\}dt_1\\
&+\frac{i\kappa_4}{nb_n}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{\beta_{jk}^3(U_{jj}*U_{kk}+U_{jk}*U_{jk})(t_1)xe_n(x)\int_{-\infty}^{\infty}t_2(U_{jk}*U_{jk})(t_2)\hat{\varphi}(t_2)dt_2\}dt_1\\
&+\frac{i\kappa_4}{nb_n}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{\beta_{jk}^3(U_{jk}*U_{jk})(t_1)xe_n(x)\int_{-\infty}^{\infty}t_2(U_{jj}*U_{kk}+U_{jk}*U_{jk})(t_2)\hat{\varphi}(t_2)dt_2\}dt_1\\
&+\frac{i\kappa_4}{3nb_n}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{\beta_{jk}^3U_{jk}(t_1)xe_n(x)\int_{-\infty}^{\infty}t_2(6U_{jj}*U_{jk}*U_{kk}+2U_{jk}*U_{jk}*U_{jk})(t_2)\hat{\varphi}(t_2)dt_2\}dt_1\\
&-\frac{2\kappa_4}{\sqrt{n^3b_n}}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{\beta_{jk}^3(U_{jj}*U_{kk}+U_{jk}*U_{jk})(t_1)x^2e_n(x)(\int_{-\infty}^{\infty}t_2U_{jk}(t_2)\hat{\varphi}(t_2)dt_2)^2\}dt_1\\
&-\frac{2\kappa_4}{\sqrt{n^3b_n}}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{\beta_{jk}^3U_{jk}(t_1)x^2e_n(x)\int_{-\infty}^{\infty}t_2U_{jk}(t_2)\hat{\varphi}(t_2)dt_2\int_{-\infty}^{\infty}t_3(U_{jj}*U_{kk}+U_{jk}*U_{jk})(t_3)\hat{\varphi}(t_3)dt_3\}dt_1\\
&-\frac{4i\kappa_4}{3n^2}\int_0^t\sum_{(j,k)\in I_n}\mathbb{E}\{\beta_{jk}^3U_{jk}(t_1)x^3e_n(x)(\int_{-\infty}^{\infty}t_2U_{jk}(t_2)\hat{\varphi}(t_2)dt_2)^3\}dt_1.
\end{aligned} \tag{3.108}$$

Thus, $T_3=T_{31}+T_{32}+T_3'$. Since we have already bounded T_3' in (3.105), we are left with estimating the first two terms in the sum. There are two types of sums over $(j,k)\in I_n$ in (3.108), namely the first one corresponding to $U_{jj}U_{jk}U_{jk}U_{kk}$ and the second one to $U_{jk}U_{jk}U_{jk}U_{jk}$. Define

$$J_1(s_1,s_2,s_3,s_4)=\sum_{(j,k)\in I_n}U_{jj}(s_1)U_{jk}(s_2)U_{jk}(s_3)U_{kk}(s_4), \tag{3.109}$$

$$J_2(s_1,s_2,s_3,s_4)=\sum_{(j,k)\in I_n}U_{jk}(s_1)U_{jk}(s_2)U_{jk}(s_3)U_{jk}(s_4). \tag{3.110}$$

We note that

$$|J_1|\le\sum_{(j,k)\in I_n}|U_{jk}(s_2)U_{jk}(s_3)|\le\left\{\sum_{j,k=1}^n|U_{jk}(s_2)|^2\sum_{j,k=1}^n|U_{jk}(s_3)|^2\right\}^{1/2}=n. \tag{3.111}$$

Similarly,

$$|S_2|\le n. \tag{3.112}$$

It follows from the last two inequalities and (3.1) that

$$|T_{32}(x,t)|\le\sqrt{\frac{n}{b_n^3}}C_4((b_n/n)^{1/2}x,t). \tag{3.113}$$

Now, we estimate T_{31}. Recall that T_{31} is defined in (3.106-3.107). Let us denote

$$v_n(s_1, s_2, s_3, s_4) = (nb_n)^{-1} \sum_{(j,k)\in I_n} U_{jj}(s_1)U_{jj}(s_2)U_{kk}(s_3)U_{kk}(s_4), \tag{3.114}$$

and

$$\bar{v}_n(s_1, s_2, s_3, s_4) := \mathbb{E}\{v_n(s_1, s_2, s_3, s_4)\}. \tag{3.115}$$

The rest of the proof of Proposition 3.6 follows from the next two lemmas.

Lemma 3.7.

$$\bar{v}_n(s_1, s_2, s_3, s_4) = 2\bar{v}_n(s_1)\bar{v}_n(s_2)\bar{v}_n(s_3)\bar{v}_n(s_4) + h(s_1, s_2, s_3, s_4), \tag{3.116}$$

where $\bar{v}_n(\cdot)$ *is given by (3.30) and*

$$|h(s_1, s_2, s_3, s_4)| \le \frac{6(|s_1| + |s_2| + |s_3|)}{\sqrt{mb_n}} + \frac{1}{b_n}. \tag{3.117}$$

Proof.

$$\begin{aligned}\mathbb{E}\{U_{jj}(s_1)U_{jj}(s_2)U_{kk}(s_3)U_{kk}(s_4)\} &= \mathbb{E}\{U^\circ_{jj}(s_1)U_{jj}(s_2)U_{kk}(s_3)U_{kk}(s_4)\} + \mathbb{E}\{U_{jj}(s_1)\}\mathbb{E}\{U^\circ_{jj}(s_2)U_{kk}(s_3)U_{kk}(s_4)\}\\ &+\mathbb{E}\{U_{jj}(s_1)\}\mathbb{E}\{U_{jj}(s_2)\}\mathbb{E}\{U^\circ_{kk}(s_3)U_{kk}(s_4)\} + \mathbb{E}\{U_{jj}(s_1)\}\mathbb{E}\{U_{jj}(s_2)\}\mathbb{E}\{U_{kk}(s_3)\}\mathbb{E}\{U_{kk}(s_4)\}.\end{aligned} \tag{3.118}$$

It follows from (3.68) that

$$\begin{aligned}&|\mathbb{E}\{U_{jj}(s_1)U_{jj}(s_2)U_{kk}(s_3)U_{kk}(s_4)\} - \mathbb{E}\{U_{jj}(s_1)\}\mathbb{E}\{U_{jj}(s_2)\}\mathbb{E}\{U_{kk}(s_3)\}\mathbb{E}\{U_{kk}(s_4)\}|\\ &\le Var^{1/2}\{U_{jj}(s_1)\} + Var^{1/2}\{U_{jj}(s_2)\} + Var^{1/2}\{U_{kk}(s_3)\} \le \frac{2(|s_1| + |s_2| + |s_3|)}{\sqrt{mb_n}}.\end{aligned} \tag{3.119}$$

Therefore, we obtain

$$|\bar{v}_n(s_1, s_2, s_3, s_4) - (nb_n)^{-1} \sum_{(j,k)\in I_n} \mathbb{E}\{U_{jj}(s_1)\}\mathbb{E}\{U_{jj}(s_2)\}\mathbb{E}\{U_{kk}(s_3)\}\mathbb{E}\{U_{kk}(s_4)\}| \le \frac{6(|s_1| + |s_2| + |s_3|)}{\sqrt{mb_n}}. \tag{3.120}$$

In addition,

$$(nb_n)^{-1} \sum_{(j,k)\in I_n} \mathbb{E}\{U_{jj}(a)\}\mathbb{E}\{U_{jj}(b)\}\mathbb{E}\{U_{kk}(c)\}\mathbb{E}\{U_{kk}(d)\} = (2 + 1/b_n)\bar{v}_n(a)\bar{v}_n(b)\bar{v}_n(c)\bar{v}_n(d) \tag{3.121}$$

Now the lemma follows from (3.120-3.121) and $\bar{v}_n(t) \le 1$. □

The next lemma deals with T_{31} defined in (3.106-3.107).

Lemma 3.8.

$$T_{31} = 2i\kappa_4 x Z_n(x) \int_0^t \int_{-\infty}^{\infty} t_2 \bar{v}_n * \bar{v}_n(t_1)\bar{v}_n * \bar{v}_n(t_2)\hat{\varphi}(t_2)dt_2 dt_1 + \delta_n(x, t), \tag{3.122}$$

where

$$\delta_n(x, t) \to 0 \tag{3.123}$$

uniformly on any bounded subset of $\{(x, t), t \ge 0\}$.

Proof. T_{31} can be written as

$$\begin{aligned}
T_{31}(x,t) &= \frac{\kappa_4}{\sqrt{nb_n^3}}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{(U_{jj}*U_{jj}*U_{kk}*U_{kk})(t_1)e_n^\circ(x)\}dt_1 \\
&\quad +\frac{i\kappa_4}{nb_n}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{(U_{jj}*U_{kk})(t_1)xe_n^\circ(x)\int_{-\infty}^{\infty} t_2(U_{jj}*U_{kk})(t_2)\hat{\varphi}(t_2)dt_2\}dt_1 \\
&\quad +\frac{i\kappa_4}{nb_n}\int_0^t \sum_{(j,k)\in I_n} \mathbb{E}\{(U_{jj}*U_{kk})(t_1)x\mathbb{E}\{e_n(x)\}\int_{-\infty}^{\infty} t_2(U_{jj}*U_{kk})(t_2)\hat{\varphi}(t_2)dt_2\}dt_1 \\
&\quad +T_{31}'(x,t) \\
&=: T_{311}+T_{312}+T_{313}+T_{31}'(x,t). \qquad (3.124)
\end{aligned}$$

where T_{31}' comes from the diagonal terms, so $|T_{31}'| \le \sqrt{\frac{n}{b_n^3}}C_4(\sqrt{b_n/n}x,t)$. Then

$$T_{311} = \frac{\kappa_4\sqrt{n}}{\sqrt{b_n}}\int_0^t\int_0^{t_1}\int_0^{t_2}\int_0^{t_3} \mathbb{E}\{v_n(t_1-t_2,t_2-t_3,t_3-t_4,t_4)e_n^\circ(x)\}dt_4dt_3dt_2dt_1, \qquad (3.125)$$

$$T_{312} = i\kappa_4 x\int_0^t\int_{-\infty}^{\infty}\int_0^{t_1}\int_0^{t_2} t_2\mathbb{E}\{v_n(t_3,t_4,t_1-t_3,t_2-t_4)e_n^\circ(x)\}\hat{\varphi}(t_2)dt_4dt_3dt_2dt_1, \qquad (3.126)$$

$$T_{313} = i\kappa_4 xZ_n(x)\int_0^t\int_{-\infty}^{\infty}\int_0^{t_1}\int_0^{t_2} t_2\bar{v}_n(t_3,t_4,t_1-t_3,t_2-t_4)\hat{\varphi}(t_2)dt_4dt_3dt_2dt_1. \qquad (3.127)$$

Moreover, by Lemma 3.8,

$$\begin{aligned}
T_{313} &= 2i\kappa_4 xZ_n(x)\int_0^t\int_{-\infty}^{\infty}\int_0^{t_1}\int_0^{t_2} t_2\bar{v}_n(t_3)\bar{v}_n(t_4)\bar{v}_n(t_1-t_3)\bar{v}_n(t_2-t_4)\hat{\varphi}(t_2)dt_4dt_3dt_2dt_1+\tau_n(x,t) \\
&= 2i\kappa_4 xZ_n(x)\int_0^t\int_{-\infty}^{\infty} t_2\bar{v}_n*\bar{v}_n(t_1)\bar{v}_n*\bar{v}_n(t_2)\hat{\varphi}(t_2)dt_2dt_1+\tau_n(x,t), \qquad (3.128)
\end{aligned}$$

where

$$\tau_n(x,t) = i\kappa_4 xZ_n(x)\int_0^t\int_{-\infty}^{\infty}\int_0^{t_1}\int_0^{t_2} t_2h(t_3,t_4,t_1-t_3,t_2-t_4)\hat{\varphi}(t_2)dt_4dt_3dt_2dt_1.$$

Thus,

$$\begin{aligned}
|\tau_n(x,t)| &\le |\kappa_4 xZ_n(x)|\int_0^t\int_{-\infty}^{\infty}\int_0^{t_1}\left|\int_0^{t_2}|t_2h(t_3,t_4,t_1-t_3,t_2-t_4)|\hat{\varphi}(t_2)dt_4\right|dt_3dt_2dt_1 \\
&\le \frac{|x|}{\sqrt{b_n}}C(t^4+1). \qquad (3.129)
\end{aligned}$$

Since $|e_n^\circ(x)|\le 2$,

$$\begin{aligned}
|\mathbb{E}\{v_n(s_1,s_2,s_3,s_4)e_n^\circ\}| &\le 2\mathbb{E}\{|v_n^\circ(s_1,s_2,s_3,s_4)|\} \le \frac{2}{nb_n}\sum_{(j,k)\in I_n}\mathbb{E}\{|(U_{jj}(s_1)U_{jj}(s_2)U_{kk}(s_3)U_{kk}(s_4))^\circ|\} \\
&\le \frac{4}{nb_n}\sum_{(j,k)\in I_n}\mathbb{E}\{|(U_{jj}(s_1)U_{jj}(s_2))^\circ|\}+\mathbb{E}\{|(U_{kk}(s_3)U_{kk}(s_4))^\circ|\}. \qquad (3.130)
\end{aligned}$$

Also,

$$\begin{aligned}
Var\{U_{jj}(s_1)U_{jj}(s_2)\} &\le \frac{4}{mb_n}\sum_{(p,q)\in I_n^+}\mathbb{E}\{|U_{jp}*U_{jq}(s_1)U_{jj}(s_2)+U_{jj}(s_1)U_{jp}*U_{jq}(s_2)|^2\} \\
&\le \frac{8}{mb_n}\sum_{(p,q)\in I_n}\mathbb{E}\{|U_{jp}*U_{jq}(s_1)U_{jj}(s_2)|^2\}+\mathbb{E}\{U_{jj}(s_1)U_{jp}*U_{jq}(s_2)|^2\}. \qquad (3.131)
\end{aligned}$$

and

$$\sum_{(p,q)\in I_n} \mathbb{E}\{|U_{jp} * U_{jq}(s_1)U_{jj}(s_2)|^2\} = \mathbb{E}\{\sum_{(p,q)\in I_n} |\int_0^a U_{jp}(s)U_{jq}(s_1-s)U_{jj}(s_2)ds|^2\}$$
$$\le |s_1|\mathbb{E}\{\sum_{(p,q)\in I_n} \int_0^{s_1} |U_{jp}(s)U_{jq}(s_1-s)|^2 ds\} \le s_1^2. \quad (3.132)$$

Thus,

$$Var\{U_{jj}(s_1)U_{jj}(s_2)\} \le \frac{8(s_1^2+s_2^2)}{mb_n}, \quad (3.133)$$

and we obtain

$$|\mathbb{E}\{v_n(s_1,s_2,s_3,s_4)e_n^\circ\}| \le 12\sqrt{\frac{8(s_1^2+s_2^2)}{mb_n}} + 12\sqrt{\frac{8(s_3^2+s_4^2)}{mb_n}} = \frac{C}{\sqrt{mb_n}}(\sqrt{s_1^2+s_2^2}+\sqrt{s_3^2+s_4^2}). \quad (3.134)$$

So

$$|T_{311}| \le \frac{C|\kappa_4|\sqrt{n}}{\sqrt{m}b_n}\int_0^t\int_0^{t_1}\int_0^{t_2}\int_0^{t_3} \sqrt{(t_1-t_2)^2+(t_2-t_3)^2}+\sqrt{(t_3-t_4)^2+t_4^2}dt_4dt_3dt_2dt_1 \le \frac{C\sqrt{n}}{b_n}|t|^5. \quad (3.135)$$

and

$$|T_{312}| \le \frac{C|\kappa_4 x|}{\sqrt{mb_n}}\int_0^t\int_{-\infty}^{\infty}\int_0^{t_1}|\int_0^{t_2}|t_2|(\sqrt{t_3^2+t_4^2}+\sqrt{(t_1-t_3)^2+(t_2-t_4)^2})|\hat{\varphi}(t_2)|dt_4|dt_3dt_2dt_1 \le \frac{|x|}{\sqrt{b_n}}C(|t|^3+1). \quad (3.136)$$

Since $n/b_n^2 \to 0$, we observe that $\delta = T_{311}+T_{312}+\tau_n+T_{31}'$ goes to zero uniformly on any bounded subset of $\{t \ge 0, x \in \mathbb{R}\}$. This finishes the proof of Lemma 3.8. □

Now, we are ready to finish the proof of Proposition 3.6. Indeed,

$$T_3 = T_{31}+T_{32}+T_3'$$
$$= 2i\kappa_4 x Z_n(x)\int_0^t\int_{-\infty}^{\infty} t_2\bar{v}_n*\bar{v}_n(t_1)\bar{v}_n*\bar{v}_n(t_2)\hat{\varphi}(t_2)dt_2dt_1 + \delta_n(x,t)+T_{32}+T_3'. \quad (3.137)$$

The statement of Proposition 3.6 now follows from (3.105), (3.113), and (3.123). □

To finish the proof of Proposition 3.1, we observe that the equation (3.28) follows from (3.46), (3.49), and Propositions 3.3-3.6. Proposition 3.1 is proven. □

3.5 The limit of A_n

In this subsection, we study the limit of $A_n(t)$ as $n \to \infty$. This, in turn, will allow us to study the limiting behavior of $Y_n(x,t)$. The main result of subsection is the following proposition.

Proposition 3.9. *Let $A_n(t)$ be as defined in (3.29). Then the limit of $A_n(t)$ as $n \to \infty$ exists and*

$$A(t) := \lim_{n\to\infty} A_n(t) \quad (3.138)$$
$$= -2\sigma^2\int_0^t \frac{1}{8\pi^3\sigma^2}\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma}\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} e^{it_1x}\varphi'(y)\sqrt{8\sigma^2-x^2}\sqrt{8\sigma^2-y^2}F_\sigma(x,y)1_{\{x\ne y\}}dxdydt_1,$$

where for $x \neq y$

$$F_\sigma(x,y) = \frac{\pi}{2\sigma^2}\sum_{k=0}^{\infty} U_k(x)U_k(y)\gamma_k \tag{3.139}$$

$$= \int_{-\infty}^{\infty} \frac{\frac{\sin^3 s}{s^3} - \frac{\sin s}{s}}{2\sigma^2\left(1-\frac{\sin^2 s}{s^2}\right)^2 - \left(\frac{\sin s}{s}+\frac{\sin^3 s}{s^3}\right)xy + \frac{\sin^2 s}{s^2}(x^2+y^2)} ds. \tag{3.140}$$

Proof. We recall that $A_n(t)$ is defined in (3.29) as $A_n(t) = -\frac{2\sigma^2}{n}\int_0^t \sum_{(j,k)\in I_n}\mathbb{E}\{U_{jk}(t_1)\varphi'(M)_{jk}\}dt_1$. In the full Wigner matrix case, one has $A_n = -\frac{2\sigma^2}{n}\int_0^t Tr[e^{it_1 M}\varphi(M)]dt_1$, and the limiting behavior of A_n immediately follows from the Wigner semi-circle law. In the band matrix case, there are additional difficulties due to the fact that the summation in the formula for A_n is restricted to the band entries, i.e. to $(j,k)\in I_n$.

We start with the definition of a bilinear form on $C_b(\mathbb{R})$, the space of bounded continuous functions on $\mathbb{R}$.

Definition 3.10. Let $f,g\in C_b(\mathbb{R})$. Define

$$< f,g >_n := n^{-1}\mathbb{E}\{\sum_{(j,k)\in I_n} f(M)_{jk}\overline{g(M)_{jk}}\}. \tag{3.141}$$

It follows from the above definition that

$$A_n(t) = -2\sigma^2\int_0^t < e^{it_1 x}, \varphi'(x) >_n dt_1. \tag{3.142}$$

The bilinear form (3.141) is an inner product (perhaps, degenerate).
(1)$< f,f >_n \geq 0$,
(2)$< f,g >_n = \overline{< g,f >_n}$,
(3)$< f,g_1+g_2 >_n = < f,g_1 >_n + < f,g_2 >_n$, $< kf,g >_n = k < f,g >_n$, $k\in\mathbb{R}$,
(4)(Cauchy-Schwarz Inequality) $| < f,g >_n | \leq < f,f >_n^{1/2} < g,g >_n^{1/2}$.

The proof of Proposition 3.9 relies on two auxiliary lemmas.

Lemma 3.11. *For all $f,g\in C_b(\mathbb{R})$ the limit*

$$< f,g > := \lim_{n\to\infty} < f,g >_n \tag{3.143}$$

exists.

Proof. We start with monomials. While monomials do not belong to $C_b(\mathbb{R})$, the expression (3.141) still makes sense since all moments of the matrix entries of M are finite. For $l,m\in\mathbb{N}$, consider $f(x)=x^l,\ g(x)=x^m$. Then

$$< x^l, x^m >_n = \frac{1}{nb_n^{(l+m)/2}} \sum_{(i_0,i_1),\ldots,(i_{l+m-1},i_0),(i_l,i_0)\in I_n} \mathbb{E}\{W_{i_0 i_1}\ldots W_{i_{l+m-1}i_0}\}. \tag{3.144}$$

Let us fix $i_0\in\{1,\ldots,n\}$. For $k=1,\ldots,l+m$, define

$$x_k = \begin{cases} i_k - i_{k-1} & \text{if } |i_k-i_{k-1}|\leq b_n, \\ i_k-i_{k-1}-n & \text{if } i_k-i_{k-1} > b_n, \\ n+(i_k-i_{k-1}) & \text{if } i_k-i_{k-1} < -b_n, \end{cases} \tag{3.145}$$

where $i_{l+m}=i_0$. Since l,m are fixed and $n/b_n\to\infty$, for sufficiently large n the restriction (i_0,i_1), ..., (i_{l+m-1},i_0), $(i_l,i_0)\in I_n$ is equivalent to $|x_1|,\ldots,|x_{l+m}|\leq b_n,\ x_1+\ldots+x_{l+m}=0$, and $|x_1+\ldots+x_l|\leq b_n$.

Therefore, for sufficiently large n,

$$< x^l, x^m >_n = \frac{1}{n b_n^{(l+m)/2}} \sum_{i_0=1}^{n} \sum_{\substack{|x_1|,...,|x_{l+m}| \le b_n \\ x_1+x_2+...+x_{l+m}=0 \\ |x_1+x_2+...+x_l| \le b_n}} \mathbb{E}\{W_{i_0,i_1}...W_{i_{l+m-1},i_0}\}. \tag{3.146}$$

Each $(i_0, i_1, ..., i_{l+m-1}, i_0)$ is a closed path such that the distance between the endpoints of each edge is bounded by b_n, and, in addition, the distance between i_0 and i_l is also bounded by b_n. If $l+m$ is odd, one can show that $< x^l, x^m >_n \to 0$ using power counting and independence of matrix entries. The proof is very similar to the combinatorial argument in the proof of the Wigner semicircle law and is left to the reader.

Now consider the case when $l+m$ is even. Without loss of generality we can assume that $l \le m$. As in the proof of the semicircle law, only the paths where every edge appears exactly twice contribute to the limit. For each such path,

$$\mathbb{E}\{W_{i_0,i_1}...W_{i_{l+m-1},i_0}\} = \sigma^{l+m}.$$

Moreover, each such $(i_0, i_1, ..., i_{l+m-1}, i_0)$ corresponds to a Dyck path of length $l+m$ (see e.g. [1]). Recall that a Dyck path $(s(0), ..., s(l+m))$ of length $m+l$ satisfies

$$s(0) = s(l+m) = 0, \ s(1), ..., s(l+m-1) \ge 0, \ \text{and} \ |s(t+1) - s(t)| = 1, \ i = 0, ..., l+m-1.$$

Specifically, $s(t+1) - s(t) = 1$ if the non-oriented edge (i_t, i_{t+1}) appears in $(i_0, i_1, ..., i_{l+m-1}, i_0)$ for the first time and $s(t+1) - s(t) = -1$ if the edge (i_t, i_{t+1}) appears in $(i_0, i_1, ..., i_{l+m-1}, i_0)$ for the second time.

If one removes in (3.144) the condition that $(i_l, i_0) \in I_n$ then the l.h.s. in (3.144) becomes $\frac{1}{n} Tr M^{l+m}$ and each Dyck path gives equal contribution in the limit $n \to \infty$. However, we have to take into account the condition $(i_l, i_0) \in I_n$. As a result, the combinatorial analysis becomes more involved. Suppose $s(l) = k, \ 0 \le k \le l$. Then during the first l steps of the path $(i_0, i_1, ..., i_{l+m-1}, i_0)$, $(l-k)/2$ edges appear twice and k edges appear only once. For each of the edges appearing twice, the corresponding two numbers x_i have the same absolute value but differ in sign. The remaining k numbers x_i will be renumerated (in the order of their appearance) by $y_1, y_2, \dots y_k$. One obtains

$$\begin{aligned} < x^l, x^m >_n = & \frac{\sigma^{l+m}}{b_n^{l+m}} \sum_{k=0}^{l} \#\{Dyck\ paths\ of\ length\ l+m\ with\ s(l)=k\} \\ & \times \#\text{of integeres}\{|y_1| \le b_n, \dots, |y_k| \le b_n, \dots, |y_{l+m}| \le b_n, |y_1+...+y_k| \le b_n\} \\ & + O(b_n^{-1}). \end{aligned} \tag{3.147}$$

Therefore, $< x^l, x^m > = \lim_{n\to\infty} < x^l, x^m >_n$ exists, and

$$\begin{aligned} < x^l, x^m > = & (\sqrt{2}\sigma)^{l+m} \sum_{k=0}^{l} \#\{Dyck\ paths\ of\ length\ l+m\ with\ s(l)=k\} \\ & \times Vol\{|t_1| \le 1/2, |t_2| \le 1/2, \dots, |t_{(l+m)/2}| \le 1/2, |t_1+t_2+...+t_k| \le 1/2\}\}. \end{aligned} \tag{3.148}$$

The number of Dyck paths with $s(l) = k$ is

$$\left[\binom{l}{\frac{l+k}{2}} - \binom{l}{\frac{l+k+2}{2}}\right] \left[\binom{m}{\frac{m+k}{2}} - \binom{m}{\frac{m+k+2}{2}}\right] = \frac{(k+1)^2}{(l+1)(m+1)} \binom{l+1}{\frac{l+k+2}{2}} \binom{m+1}{\frac{m+k+2}{2}}. \tag{3.149}$$

Let $T_1, ..., T_{(l+m)/2}$ be i.i.d random variables uniformly distributed on $[-1/2, 1/2]$. Then

$$Vol\{|t_1| \le 1/2, ..., |t_k| \le 1/2, |t_1+t_2+...+t_k| \le 1/2\} = \mathbb{P}(|T_1+T_2+...+T_k| \le 1/2). \tag{3.150}$$

Let $S_k = T_1 + ... + T_k$. Then the characteristic function of S_k is

$$\mathbb{E}\{e^{ixS_k}\} = (\mathbb{E}\{e^{ixY_1}\})^k = \left(\frac{\sin x/2}{x/2}\right)^k. \tag{3.151}$$

Hence, the density function of S_k is given by

$$f_k(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixs}\left(\frac{\sin x/2}{x/2}\right)^k dx. \tag{3.152}$$

Define $\gamma_k := \mathbb{P}(|S_k| \leq 1/2)$. Then

$$\gamma_k = \int_{-1/2}^{1/2} f_k(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin x/2}{x/2}\right)^{k+1} dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^{k+1} dx = f_{k+1}(0). \tag{3.153}$$

The exact formula for γ_k is well known (see e.g. [16]):

$$\gamma_k = \begin{cases} [(2t)!]^{-1}\sum_{s=0}^{t}(-1)^s\binom{2t+1}{s}(t-s+1/2)^{2t}, & \text{if } k = 2t, \\ [(2t+1)!]^{-1}\sum_{s=0}^{t}(-1)^s\binom{2t+2}{s}(t-s+1)^{2t+1}, & \text{if } k = 2t+1. \end{cases} \tag{3.154}$$

Therefore, we conclude that

$$< x^l, x^m >= (\sqrt{2}\sigma)^{m+l} C_{l,m}, \tag{3.155}$$

where $C_{l,m}$ is defined in the following way. For $l+m$ is odd, $C_{l,m} = 0$. For $l+m$ is even, $l \leq m$,

$$C_{l,m} = \begin{cases} [(l+1)(m+1)]^{-1}\sum_{k=0}^{l/2}(2k+1)^2\binom{l+1}{\frac{l-2k}{2}}\binom{m+1}{\frac{m-2k}{2}}\gamma_{2k}, & \text{if } l \text{ is even}, \\ [(l+1)(m+1)]^{-1}\sum_{k=0}^{(l-1)/2}(2k+2)^2\binom{l+1}{\frac{l-2k-1}{2}}\binom{m+1}{\frac{m-2k-1}{2}}\gamma_{2k+1}, & \text{if } l \text{ is odd}. \end{cases} \tag{3.156}$$

For $l+m$ is even, $l > m$, $C_{l,m} = C_{m,l}$. It follows from the definition that $0 \leq C_{l,m} \leq C_{\frac{l+m}{2}}$, where $C_s = \frac{(2s)!}{s!(s+1)!}$ is the Catalan number.

If f, g are polynomials, $f(x) = \sum_{i=0}^{p} a_i x^i, g(x) = \sum_{j=0}^{q} b_j x^j$, then by linearity

$$< f, g >= \sum_{i=0}^{p}\sum_{j=0}^{q} a_i b_j (\sqrt{2}\sigma)^{i+j} C_{i,j}. \tag{3.157}$$

Thus, the result of Lemma 3.11 holds when f and g are arbitrary polynomials.

For general bounded continuous functions f, g, we will show that $\{< f, g >_n\}$ is a Cauchy sequence. To this end, we choose a sufficiently large B independent of n (it will be enough to take $B = 4\sigma^3 + 1$). Fix $\delta > 0$. By the Stone-Weierstrass theorem, there exist polynomials f_δ, g_δ such that

$$\sup_{x:|x|\leq B+1} |f(x) - f_\delta(x)| \leq \delta, \quad \sup_{x:|x|\leq B+1} |g(x) - g_\delta(x)| \leq \delta. \tag{3.158}$$

Let h be an infinitely differentiable function such that $|h| \leq 1$, $h(x) = 1$ for $|x| \leq B$, $h(x) = 0$ for $|x| \geq B+1$. We write

$$< f, g >_n = < f - f_\delta h, g - g_\delta h >_n + < f - f_\delta h, g_\delta h >_n + < f_\delta, g - g_\delta h >_n + < f_\delta h, g_\delta h >_n . \tag{3.159}$$

Below we show that the first three terms on the r.h.s. of (3.159) are small provided δ is small. It follows from (3.141) that

$$< (f - f_\delta)h, (f - f_\delta)h >_n \leq \delta^2, \tag{3.160}$$

$$< (g - g_\delta)h, (g - g_\delta)h >_n \leq \delta^2. \tag{3.161}$$

Since f,g are bounded on $\mathbb{R}$ and f_δ, g_δ are polynomials, there exists sufficiently large $N \in \mathbb{Z}_+$, such that $(f-f_\delta)^2(1-h) \le x^{2N}(1-h)$, and $(g-g_\delta)^2(1-h) \le x^{2N}(1-h)$. Then

$$\begin{aligned} &< (f-f_\delta)(1-h), (f-f_\delta)(1-h) >_n \le < x^{2N}(1-h), x^{2N}(1-h) >_n \\ &\le \mathbb{E}\frac{1}{nB^{2N}} TrM^{6N} \le \delta^2 \end{aligned} \tag{3.162}$$

for sufficiently large n, where the last inequality follows from the semicircle law provided N is chosen so that $\frac{(\sqrt{2}\sigma)^{6N}}{B^{2N}} < \delta^2$. In a similar fashion,

$$< (g-g_\delta)(1-h), (g-g_\delta)(1-h) >_n \le \delta^2, \tag{3.163}$$

$$< f(1-h), f(1-h) >_n \le \delta^2, \quad < g(1-h), g(1-h) >_n \le \delta^2. \tag{3.164}$$

for sufficiently large n. The bounds (3.160-3.164) imply

$$< f - f_\delta h, f - f_\delta h >_n \le const\delta^2, \tag{3.165}$$

$$< g - g_\delta h, g - g_\delta h >_n \le const\delta^2. \tag{3.166}$$

Now, applying (3.165-3.166) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &| < f,g >_n - < f_\delta h, g_\delta h >_n | \le Const\delta, \\ &| < f_\delta, g_\delta >_n - < f_\delta h, g_\delta h >_n | \le Const\delta, \end{aligned}$$

and, as a result,

$$| < f,g >_n - < f_\delta, g_\delta >_n | \le 2Const\delta. \tag{3.167}$$

Therefore $< f,g >_n$ is a Cauchy sequence and $< f,g >$ exists. □

In the next lemma, we diagonalize the bilinear form $< f,g >$.

Lemma 3.12. *Let $\{U_n(x)\}$ be the (rescaled) Chebyshev polynomials of the second kind on $[-2\sqrt{2}\sigma, 2\sqrt{2}\sigma]$,*

$$U_n^\sigma(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \left(\frac{x}{\sqrt{2}\sigma}\right)^{n-2k}. \tag{3.168}$$

Then $\{U_n^\sigma(x)\}_{n\ge 0}$ are orthogonal with respect to the bilinear form (3.155), i.e.

$$< U_n, U_m >= \delta_{nm}\gamma_n, \tag{3.169}$$

where γ_n is given by (3.154).

Remark 3.13. *Note that $< f_1, g_1 >=< f_2, g_2 >$ if $f_1 = f_2$ and $g_1 = g_2$ in a neighborhood of $[-2\sqrt{2}\sigma, 2\sqrt{2}\sigma]$. Thus, one can reformulate Lemma 3.12 in such a way that $\{hU_n^\sigma\}_{n\ge 0}$ are orthogonal with respect to the bilinear form (3.155).*

Remark 3.14.

Recall that the rescaled Chebyshev polynomials are orthonormal with respect to the Wigner semicircle law, i.e.

$$\int_{-2\sqrt{2}\sigma}^{-2\sqrt{2}\sigma} U_n^\sigma(x) U_m^\sigma(x) \frac{1}{4\pi\sigma^2}\sqrt{8\sigma^2 - x^2} = \delta_{nm}. \tag{3.170}$$

Also,

$$U_n^\sigma(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}, \quad x = 2\sqrt{2}\sigma\cos\theta. \tag{3.171}$$

When it does not lead to ambiguity, we will omit the super-index in the notation for the rescaled Chebyshev polynomials (alternatively, the reader can assume that $2\sqrt{2}\sigma = 1$).

Proof. Since $< x^l, x^m >= 0$ if $l+m$ is odd, it follows by linearity that

$$< U_n, U_m >= 0, \text{ if } n+m \text{ is odd.} \tag{3.172}$$

We are left to compute $< U_{2n}, U_{2m} >$ and $< U_{2n+1}, U_{2m+1} >$. We first compute $< x^{2l}, U_{2n} >$ and $< x^{2l+1}, U_{2n+1} >$ for $l = 0, 1, ..., n$. One has

$$\begin{aligned}
< x^{2l}, U_{2n} > &= (\sqrt{2}\sigma)^{2l} \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} C_{2l,2n-2k} \\
&= \frac{(\sqrt{2}\sigma)^{2l}}{2l+1} \left[\sum_{k=0}^{n-l} \frac{(-1)^k}{2n-2k+1} \binom{2n-k}{k} \sum_{t=0}^{l} (2t+1)^2 \binom{2l+1}{l-t} \binom{2n-2k+1}{n-k-t} \gamma_{2t} \right. \\
&\quad \left. + \sum_{k=n-l+1}^{n} \frac{(-1)^k}{2n-2k+1} \binom{2n-k}{k} \sum_{t=0}^{n-k} (2t+1)^2 \binom{2l+1}{l-t} \binom{2n-2k+1}{n-k-t} \gamma_{2t} \right] \\
&= \frac{(\sqrt{2}\sigma)^{2l}}{2l+1} \sum_{t=0}^{l} (2t+1)^2 \binom{2l+1}{l-t} \left[\sum_{k=0}^{n-t} \frac{(-1)^k (2n-k)!}{k!(n-k-t)!(n-k+t+1)!} \right] \gamma_{2t},
\end{aligned} \tag{3.173}$$

and

$$\begin{aligned}
< x^{2l+1}, U_{2n+1} > &= (\sqrt{2}\sigma)^{2l+1} \sum_{k=0}^{n} (-1)^k \binom{2n+1-k}{k} C_{2l+1,2n+1-2k} \\
&= \frac{(\sqrt{2}\sigma)^{2l+1}}{2l+2} \left[\sum_{k=0}^{n-l} \frac{(-1)^k}{2n-2k+2} \binom{2n+1-k}{k} \sum_{t=0}^{l} (2t+2)^2 \binom{2l+2}{l-t} \binom{2n-2k+2}{n-k-t} \gamma_{2t+1} \right. \\
&\quad \left. + \sum_{k=n-l+1}^{n} \frac{(-1)^k}{2n-2k+2} \binom{2n+1-k}{k} \sum_{t=0}^{n-k} (2t+2)^2 \binom{2l+2}{l-t} \binom{2n-2k+2}{n-k-t} \gamma_{2t+1} \right] \\
&= \frac{(\sqrt{2}\sigma)^{2l+1}}{2l+2} \sum_{t=0}^{l} (2t+2)^2 \binom{2l+2}{l-t} \left[\sum_{k=0}^{n-t} \frac{(-1)^k (2n+1-k)!}{k!(n-k-t)!(n-k+t+2)!} \right] \gamma_{2t+1}.
\end{aligned} \tag{3.174}$$

Denote

$$G_1(n,t) = \sum_{k=0}^{n-t} \frac{(-1)^k (2n-k)!}{k!(n-k-t)!(n-k+t+1)!}, \tag{3.175}$$

$$G_2(n,t) = \sum_{k=0}^{n-t} \frac{(-1)^k (2n+1-k)!}{k!(n-k-t)!(n-k+t+2)!}. \tag{3.176}$$

Then

$$< x^{2l}, U_{2n} >= \frac{(\sqrt{2}\sigma)^{2l}}{2l+1} \sum_{t=0}^{l} (2t+1)^2 \binom{2l+1}{l-t} G_1(n,t) \gamma_{2t}, \tag{3.177}$$

$$< x^{2l+1}, U_{2n+1} >= \frac{(\sqrt{2}\sigma)^{2l+1}}{2l+2} \sum_{t=0}^{l} (2t+2)^2 \binom{2l+2}{l-t} G_2(n,t) \gamma_{2t+1}. \tag{3.178}$$

It follows from (3.175-3.176) that

$$G_1(n,t) = \frac{(2n)!}{(n-t)!(n+t+1)!} {}_2F_1 \left(\begin{matrix} -(n-t), -(n+t+1) \\ -2n \end{matrix} ; 1 \right), \tag{3.179}$$

$$G_2(n,t) = \frac{(2n+1)!}{(n-t)!(n+t+2)!} {}_2F_1 \left(\begin{matrix} -(n-t), -(n+t+2) \\ -2n-1 \end{matrix} ; 1 \right), \tag{3.180}$$

where ${}_2F_1$ is a hypergeometric function. By the Chu-Vandermonde identity (see e.g. [3]), we have

$$ {}_2F_1\left(\begin{array}{c} -(n-t), -(n+t+1) \\ -2n \end{array} ;1 \right) = \frac{(-n+t+1)_{n-t}}{(-2n)_{n-t}}, \tag{3.181} $$

$$ {}_2F_1\left(\begin{array}{c} -(n-t), -(n+t+2) \\ -2n-1 \end{array} ;1 \right) = \frac{(-n+t+1)_{n-t}}{(-2n-1)_{n-t}}, \tag{3.182} $$

where $(a)_n = a(a+1)\cdots(a+n-1)$. Since

$$ (-n+t+1)_{n-t} = \begin{cases} 0 & \text{if } t = 0, 1, ..., n-1 \\ 1 & \text{if } t = n, \end{cases} \tag{3.183} $$

we obtain

$$ G_1(n,t) = 0,\ G_2(n,t) = 0, \text{for } t = 0, 1, ..., n-1, \tag{3.184} $$

and

$$ G_1(n,n) = \frac{1}{2n+1},\ G_2(n,n) = \frac{1}{2n+2}. \tag{3.185} $$

Therefore, for $l = 0, 1, \ldots, n-1$, each term at the r.h.s. of (3.177-3.178) is zero, and

$$ < x^{2n}, U_{2n} >= \frac{(\sqrt{2}\sigma)^{2n}}{2n+1}(2n+1)^2\binom{2n+1}{0}G_1(n,n)\gamma_{2n} = (\sqrt{2}\sigma)^{2n}\gamma_{2n}, \tag{3.186} $$

$$ < x^{2n+1}, U_{2n+1} >= \frac{(\sqrt{2}\sigma)^{2n+1}}{2n+2}(2n+2)^2\binom{2n+2}{0}G_2(n,n)\gamma_{2n+1} = (\sqrt{2}\sigma)^{2n+1}\gamma_{2n+1}. \tag{3.187} $$

Hence, for $m < n$,

$$ < U_{2m}, U_{2n} >= 0,\ < U_{2m+1}, U_{2n+1} >= 0, \tag{3.188} $$

and

$$ < U_{2n}, U_{2n} >=< \left(\frac{x}{\sqrt{2}\sigma}\right)^{2n}, U_{2n} >= \gamma_{2n}, \tag{3.189} $$

$$ < U_{2n+1}, U_{2n+1} >=< \left(\frac{x}{\sqrt{2}\sigma}\right)^{2n+1}, U_{2n+1} >= \gamma_{2n+1}. \tag{3.190} $$

Combining (3.188), (3.172), (3.189) and (3.190), we complete the proof of Lemma 3.12. □

Now, we are ready to finish the proof of Proposition 3.9. Let $f, g \in C_b(\mathbb{R})$, and

$$ f_k = \frac{1}{4\pi\sigma^2}\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} f(x)U_k(x)\sqrt{8\sigma^2 - x^2}dx,\ g_k = \frac{1}{4\pi\sigma^2}\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} g(x)U_k(x)\sqrt{8\sigma^2 - x^2}dx. \tag{3.191} $$

Then

$$ \begin{aligned} < f, g > &= \sum_{k=0}^{\infty} f_k g_k \gamma_k &(3.192)\\ &= \frac{1}{8\pi^3\sigma^2}\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma}\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} f(x)g(y)\sqrt{8\sigma^2 - x^2}\sqrt{8\sigma^2 - y^2}F(x,y)1_{\{x\neq y\}}dxdydt, &(3.193) \end{aligned} $$

where, for $x \neq y$,

$$ \begin{aligned} F_\sigma(x,y) &= \frac{\pi}{2\sigma^2}\sum_{k=0}^{\infty} U_k(x)U_k(y)\gamma_k \\ &= \int_{-\infty}^{\infty} \frac{\frac{\sin s}{s} - \frac{\sin^3 s}{s^3}}{2\sigma^2\left(1 - \frac{\sin^2 s}{s^2}\right)^2 - \left(\frac{\sin s}{s} + \frac{\sin^3 s}{s^3}\right)xy + \frac{\sin^2 s}{s^2}(x^2+y^2)} ds. &(3.194) \end{aligned} $$

Formula (3.192) follows for polynomials from (3.169) and (3.170), and then by continuity, by repeating the arguments at the end of the proof of Lemma 3.11, for general continuous bounded functions. Formula (3.194) is a straightforward consequence of the Fourier analysis. It follows from (3.171) that the r.h.s. of (3.192) can be rewritten as

$$< f, g >= -2 \sum_{l \neq 0} \hat{\alpha}_l \hat{\beta}_l \gamma_{|l|-1}, \tag{3.195}$$

where

$$\alpha(\theta) = f(2\sqrt{2}\sigma \cos\theta), \quad \beta(\theta) = g(2\sqrt{2}\sigma \cos\theta), \tag{3.196}$$

$$\hat{\alpha}_l = \frac{1}{2\pi} \int_0^{2\pi} \alpha(\theta) e^{-il\theta} d\theta, \quad \hat{\beta}_l = \frac{1}{2\pi} \int_0^{2\pi} \beta(\theta) e^{-il\theta} d\theta. \tag{3.197}$$

In particular, the trigonometric series $\sum_{l\neq 0} \gamma_{|l|-1} e^{il\theta}$ represents an L^1 function h which has $O(|\theta|^{-1/2})$ singularity near the origin. The convergence is pointwise for all $\theta \neq 0$,

$$h(\theta) = \sum_{l \neq 0} \gamma_{|l|-1} e^{il\theta}, \ \theta \neq 0,$$

$$\hat{h}_l = \gamma_{|l|-1}, \ \text{if} \ l \neq 0, \ \hat{h}_0 = 0.$$

The convolution of β and h is then a continuous function on the unit circle, and one can rewrite (3.195) in the integral form by applying the Parseval's theorem.

Finally, it follows from (3.29) and (3.193) that the limit of $A_n(x)$ exists and equals

$$A(t) = -2\sigma^2 \int_0^t < e^{it_1 x}, \varphi' > dt_1 \tag{3.198}$$

with

$$< e^{it_1 x}, \varphi' >= \frac{1}{8\pi^3\sigma^2} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} e^{it_1 x} \varphi'(y) \sqrt{8\sigma^2 - x^2} \sqrt{8\sigma^2 - y^2} F_\sigma(x, y) 1_{\{x \neq y\}} dx dy. \tag{3.199}$$

Proposition 3.9 is proven. □

3.6

The rest of the proof of Theorem 2.1 follows the steps in [25]. Using precompactness of $\{Y_n, Z_n\}_{n\geq 1}$, we consider a converging subsequence. Our goal is to show that the limit is unique. Let

$$Y_{n_j}(x, t) \to Y(x, t), \ Z_{n_j}(x) \to Z(x). \tag{3.200}$$

By Wigner semicircle law,

$$\bar{v}_n(t) \to \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{e^{ity}}{4\pi\sigma^2} \sqrt{8\sigma^2 - y^2} dy := v(t). \tag{3.201}$$

So the limit of $Y_n(x, t)$ satisfies the following equation:

$$\begin{aligned} Y(x, t) + 4\sigma^2 \int_0^t \int_0^{t_1} v(t_1 - t_2) Y(x, t_2) dt_2 dt_1 = \\ x Z(x) A(t) + 2i\kappa_4 x Z(x) \int_0^t v * v(t_1) dt_1 \int_{-\infty}^{\infty} t_2 v * v(t_2) \hat{\varphi}(t_2) dt_2, \end{aligned} \tag{3.202}$$

where $v * v$ is defined in (3.20). As in [25] (see (formulas (2.82)-(2.86) and Proposition 2.1 there), we can solve (3.202) to obtain

$$\begin{aligned} Y(x,t) &= -\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{2\sigma^2 e^{i\lambda t} x Z(x)}{\pi\sqrt{8\sigma^2-\lambda^2}} \int_0^t e^{-i\lambda t_1} < e^{it_1 x}, \varphi' > dt_1 d\lambda \\ &\quad + \frac{i\kappa_4 x Z(x) B}{4\pi\sigma^4} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{e^{it\lambda}(4\sigma^2-\lambda^2)}{\sqrt{8\sigma^2-\lambda^2}} d\lambda, \end{aligned} \tag{3.203}$$

where

$$B = \int_{-\infty}^{\infty} t_2 v * v(t_2)\hat{\varphi}(t_2) dt_2 = \frac{1}{4\pi\sigma^4} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \varphi(\mu) \frac{4\sigma^2-\mu^2}{\sqrt{8\sigma^2-\mu^2}} d\mu. \tag{3.204}$$

It then follows from (3.23) that

$$\begin{aligned} Z'(x) &= ixZ(x) \int_{-\infty}^{\infty} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \int_0^t \frac{2\sigma^2 e^{i\lambda t}\hat{\varphi}(t)}{\pi\sqrt{8\sigma^2-\lambda^2}} e^{-i\lambda t_1} < e^{it_1 x}, \varphi' > dt_1 d\lambda dt \\ &\quad - \frac{\kappa_4 x Z(x)}{16\pi^2\sigma^8} \left(\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{\varphi(\lambda)(4\sigma^2-\lambda^2)}{\sqrt{8\sigma^2-\lambda^2}} d\lambda \right)^2. \end{aligned} \tag{3.205}$$

One can rewrite the last formula in the form (3.5) with

$$\begin{aligned} Var_{band}[\varphi] &= -i \int_{-\infty}^{\infty} \hat{\varphi}(t) \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{2\sigma^2 e^{i\lambda t}}{\pi\sqrt{8\sigma^2-\lambda^2}} \int_0^t e^{-i\lambda t_1} < e^{it_1 x}, \varphi' > dt_1 d\lambda dt \\ &\quad + \frac{\kappa_4}{16\pi^2\sigma^8} \left(\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{\varphi(\lambda)(4\sigma^2-\lambda^2)}{\sqrt{8\sigma^2-\lambda^2}} d\lambda \right)^2 \\ &= \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{(\varphi(x)-\varphi(\lambda))\varphi'(y)\sqrt{8\sigma^2-x^2}\sqrt{8\sigma^2-y^2}}{4\pi^4(x-\lambda)\sqrt{8\sigma^2-\lambda^2}} F(x,y) 1_{\{x\neq y\}} dx dy d\lambda \\ &\quad + \frac{\kappa_4}{16\pi^2\sigma^8} \left(\int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{\varphi(\lambda)(4\sigma^2-\lambda^2)}{\sqrt{8\sigma^2-\lambda^2}} d\lambda \right)^2. \end{aligned} \tag{3.206}$$

This finishes the proof of Theorem 2.1 for test functions satisfying (3.1).

Now, let φ be an arbitrary function with bounded continuous derivative. It follows from Lemma C.2 that we can assume that φ has compact support inside the interval $[-10\sigma, 10\sigma]$. One then approximates φ in the $C_1([-10\sigma, 10\sigma])$ norm by functions satisfying (3.1) and uses the bound (3.10) to control the variance of the error term. Theorem 2.1 is proven.

4 Proof of Theorem 2.2

This section is devoted to the proof of Theorem 2.2. Thus, our goal is to xextend the result of Theorem 2.1 to the case of non-i.i.d. entries with uniformly bounded fifth moment. For technical reasons, we require that the fourth cumulant is zero and $\sqrt{n}\ln n \ll b_n$. First, we establish two auxiliary lemmas.

4.1

The first lemma is a simple statement about the norm of a sub-matrix of a unitary matrix.

Lemma 4.1. *Let U be an $n \times n$ unitary matrix and V be any $k \times k$ block of U. Then*

$$\|V\| \leq 1 \tag{4.1}$$

Proof. Suppose the indices of V in U are $(s, s+1, ..., s+k-1) \times (t, t+1, ..., t+k-1)$. Then

$$V = P_1 U P_2 \tag{4.2}$$

where P_1 is the orthogonal projection onto the subspace spanned by $e_s, ..., e_{s+k-1}$, and P_2 is the orthogonal projection onto the subspace spanned by $e_t, ..., e_{t+k-1}$. Then

$$\|V\| \le \|P_1\| \|U\| \|P_2\| = 1. \tag{4.3}$$

□

The second lemma gives an upper bound on the norm of a band matrix built from a unitary matrix.

Lemma 4.2. *Let U be an $n \times n$ unitary matrix. Let b be a positive integer smaller than $n/2$. Denote $I := \{(j,k)|\ j,k = 1, ..., n,\ |j-k| \le b \text{ or } n - |j-k| \le b\}$. Let*

$$U^{(band)} := \{U_{jk}, (j,k) \in I; 0 \text{ otherwise}\}_{j,k=1}^{n}. \tag{4.4}$$

Then there exist positive constants C_1 and C_2, independent from n and b, such that

$$\|U^{(band)}\| \le C_1 \ln b + C_2. \tag{4.5}$$

Proof. Define

$$A = \{U_{jk},\ |j-k| \le b\ ; 0 \text{ otherwise}\}_{j,k=1}^{n}, \tag{4.6}$$

$$B = \{U_{jk},\ n - |j-k| \le b\ ; 0 \text{ otherwise}\}_{j,k=1}^{n}. \tag{4.7}$$

Then

$$U^{(band)} = A + B, \tag{4.8}$$

and

$$\|U^{(band)}\| \le \|A\| + \|B\|. \tag{4.9}$$

Matrix B can be written as

$$B = \begin{bmatrix} 0 & 0 & B_1 \\ 0 & 0 & 0 \\ B_2 & 0 & 0 \end{bmatrix} \tag{4.10}$$

where B_1, B_2 are $(b+1) \times (b+1)$ matrices and B_1 (B_2) is a strictly upper (lower) triangular matrix obtained from the corresponding $(b+1) \times (b+1)$ block of U by making all entries below (above) the main diagonal zero. It is known (see e.g. [26]) that if B_{upper} is an upper triangular matrix constructed in such a way from an $N \times N$ matrix B then $\|B_{upper}\| \le O(\log N)\|B\|$. Applying Lemma 4.1, we obtain

$$\begin{aligned} &\|B_1\|, \|B_2\| \le Const \ln(b+1), \\ &\|B\| \le \|B_1\| + \|B_2\| \le 2Const \ln(b+1). \end{aligned} \tag{4.11}$$

Now we turn our attention to the norm of A. Write $n = m \times (b+1) - r,\ 0 \le r \le b$. Define

$$U' = \begin{bmatrix} U & 0_{n \times r} \\ 0_{r \times n} & 0_{r \times r} \end{bmatrix}, A' = \begin{bmatrix} A & 0_{n \times r} \\ 0_{r \times n} & 0_{r \times r}. \end{bmatrix} \tag{4.12}$$

Then $\|U'\| = \|U\| = 1, \|A'\| = \|A\|$ and A' can be written as a block matrix

$$\begin{bmatrix} A_{11} & A_{12} & 0 & ... & ... & ... \\ A_{21} & A_{22} & A_{23} & ... & ... & ... \\ 0 & A_{32} & A_{33} & ... & ... & ... \\ ... & ... & ... & ... & ... & 0 \\ ... & ... & ... & ... & A_{m-1,m-1} & A_{m-1,m} \\ ... & ... & ... & 0 & A_{m,m-1} & A_{m,m}, \end{bmatrix} \tag{4.13}$$

where $A'_{jj}s$ are $(b+1)\times(b+1)$ blocks of U'. Moreover, $A'_{jk}s$ $(j\neq k)$ are strictly lower or upper triangular submatrices of the corresponding blocks in U'. Again, applying the Mathias bound in [26], we have

$$\|A_{jj}\|\leq 1,\ \|A_{jk}\|\leq Const\ln(b+1). \tag{4.14}$$

Let

$$D=Diag\{A_{11},...,A_{mm}\}. \tag{4.15}$$

Then

$$\|D\|\leq \max_{1\leq i\leq m}\|A_{ii}\|\leq 1. \tag{4.16}$$

Let

$$A_i=\begin{bmatrix} 0 & A_{i,i+1}\\ A_{i+1,i} & 0\end{bmatrix}, i=1,...,m-1. \tag{4.17}$$

Then A'_is are $2(b+1)\times 2(b+1)$ matrices, and

$$\|A_i\|\leq\|A_{i,i+1}\|+\|A_{i+1,i}\|=2Const\ln(b+1). \tag{4.18}$$

If m is even, let

$$\begin{aligned}E&=Diag\{A_1,A_3,...,A_{m-1}\},\\ F&=Diag\{0_{1\times 1},A_2,A_4,...,A_{m-2},0_{1\times 1}\}.\end{aligned}$$

If m is odd, let

$$\begin{aligned}E&=Diag\{A_1,A_3,...,A_{m-2},0_{1\times 1}\},\\ F&=Diag\{0_{1\times 1},A_2,A_4,...,A_{m-1}\}.\end{aligned}$$

Then

$$A'=D+E+F,$$

and

$$\|E\|,\|F\|\leq\max_{1\leq i\leq m}\{\|A_i\|\}=2Const\ln(b+1). \tag{4.19}$$

Therefore, we have

$$\|A'\|\leq\|D\|+\|E\|+\|F\|=1+4Const\ln(b+1)\leq C\ln(b+1). \tag{4.20}$$

Therefore,

$$\|A\|\leq C\ln(b+1). \tag{4.21}$$

Finally, (4.9),(4.11), and (4.21) imply (4.5). □

4.2

Now, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let $\hat{M}=b_n^{-1/2}\hat{W}$ be a band random real symmetric matrix with independent Gaussian random variables, and M be an arbitrary band real symmetric random matrix satisfying the conditions in Theorem 2.2. We denote, respectively, by $\hat{\mathscr{M}}_n^\circ[\varphi]$ and $\mathscr{M}_n^\circ[\varphi]$ the centered normalized linear eigenvalue statistics of $\hat{M}$ and M defined as in (2.2). Since Gaussian distribution satisfies the Poincaré inequality, Theorem 2.1 establishes the Central Limit Theorem for $\hat{\mathscr{M}}_n^\circ[\varphi]$. Thus, it suffices to show that, for every $x\in\mathbb{R}$,

$$R_n(x):=\mathbb{E}\{e^{ix\mathscr{M}_n^\circ[\varphi]}\}-\mathbb{E}\{e^{ix\hat{\mathscr{M}}_n^\circ[\varphi]}\}\to 0,\ n\to\infty. \tag{4.22}$$

Let us denote

$$e_n(s,x) = \exp\{(b_n/n)^{1/2} ixTr\varphi(M(s))^\circ\}, \tag{4.23}$$

where $M(s)$ is the interpolating matrix $M(s) = s^{1/2}M + (1-s)^{1/2}\hat{M},\ 0 \le s \le 1$. We have

$$R_n(x) = \int_0^1 \frac{\partial}{\partial s}\mathbb{E}\{e_n(s,x)\}ds. \tag{4.24}$$

Taking into account that

$$\begin{aligned}
\frac{\partial}{\partial s}e_n(s,x) &= \sum_{(j,k)\in I_n^+} \frac{\partial e_n(s,x)}{\partial M_{jk}(s)}\frac{\partial M_{jk}(s)}{\partial s} \\
&= (b_n/n)^{1/2} ixe_n(s,x) \sum_{(j,k)\in I_n^+} \frac{\partial Tr\varphi(M(s))^\circ}{\partial M_{jk}(s)}\frac{\partial M_{jk}(s)}{\partial s} \\
&= (b_n/n)^{1/2} ixe_n(s,x) \sum_{(j,k)\in I_n^+} 2\beta_{jk}(\varphi'_{jk}(M(s)))^\circ \frac{1}{2}(s^{-1/2}M_{jk} - (1-s)^{-1/2}\hat{M}_{jk}) \\
&= \frac{\sqrt{b_n}}{2\sqrt{n}} ixe_n(s,x)Tr(\varphi'(M(s)))^\circ(s^{-1/2}M - (1-s)^{-1/2}\hat{M}), \qquad (4.25)
\end{aligned}$$

we can write

$$R_n(x) = \frac{ix}{2\sqrt{n}}\int_0^1 \mathbb{E}\{e_n^\circ(s,x)Tr\varphi'(M(s))(s^{-1/2}W - (1-s)^{-1/2}\hat{W})\}ds. \tag{4.26}$$

Since

$$\varphi'(M) = i\int_{-\infty}^{+\infty} \hat{\varphi}(t)tU(t)dt, \tag{4.27}$$

we can rewrite (4.26) as

$$\begin{aligned}
R_n(x) &= -\frac{x}{2\sqrt{n}}\int_0^1\int \hat{\varphi}(t)t\mathbb{E}\{e_n^\circ(s,x)TrU(s,t)(s^{-1/2}W - (1-s)^{-1/2}\hat{W})\}dtds \qquad (4.28)\\
&= -\frac{x}{2}\int_0^1\int \hat{\varphi}(t)t[K_n - L_n]dtds, \qquad (4.29)
\end{aligned}$$

where

$$\begin{aligned}
K_n &= \frac{1}{\sqrt{ns}}\sum_{(j,k)\in I_n}\mathbb{E}\{W_{jk}\Phi_n\}, \\
L_n &= \frac{1}{\sqrt{n(1-s)}}\sum_{(j,k)\in I_n}\mathbb{E}\{\hat{W}_{jk}\Phi_n\}, \quad \text{and} \\
\Phi_n &= U_{jk}(s,t)e_n^\circ(s,x), \quad U(s,t) = e^{itM(s)}.
\end{aligned}$$

Applying the decoupling formula with $p = 3$ to every term in K_n and L_n, we obtain

$$K_n - L_n = I_2 + I_3 + \varepsilon_3, \tag{4.30}$$

where

$$I_l = \frac{s^{(l-1)/2}}{l! n^{1/2} b_n^{l/2}}\sum_{(j,k)\in I_n}\kappa_{l+1,jk}\mathbb{E}\{D_{jk}^l(s)\Phi_n\}, D_{jk}(s)\partial/\partial M_{jk}(s),\ l = 2,3, \tag{4.31}$$

and

$$|\varepsilon_3| \le \frac{C_3\sigma_5}{\sqrt{n}b_n^2}\sum_{(j,k)\in I_n}\sup_{M\in\mathbb{R}}|D_{jk}^4(s)\Phi_n|_{M(s)=M}|. \tag{4.32}$$

Let us consider I_2 first.

$$
\begin{aligned}
I_2 \;\;=\;\; & \frac{\sqrt{s}\kappa_3}{n^{3/2}} 2x^2 \sum_{(j,k)\in I_n} \beta_{jk}^2 \mathbb{E}\{e_n(s,x)U_{jk}(s,t)\left[\int \theta\hat{\varphi}(\theta)U_{jk}(s,\theta)d\theta\right]^2\} \\
& -\frac{\sqrt{s}\kappa_3}{n\sqrt{b_n}} ix \sum_{(j,k)\in I_n} \beta_{jk}^2 \mathbb{E}\{e_n(s,x)U_{jk}(s,t)\int \theta\hat{\varphi}(\theta)[U_{jj}*U_{kk}+U_{jk}*U_{jk}](s,\theta)d\theta\} \\
& -\frac{\sqrt{s}\kappa_3}{n\sqrt{b_n}} x \sum_{(j,k)\in I_n} \beta_{jk}^2 \mathbb{E}\{e_n(s,x)[U_{jj}*U_{kk}+U_{jk}*U_{jk}](s,t)\int \theta\hat{\varphi}(\theta)U_{jk}(s,\theta)d\theta\} \\
& -\frac{\sqrt{s}\kappa_3}{\sqrt{n}b_n} \sum_{(j,k)\in I_n} \beta_{jk}^2 \mathbb{E}\{e_n^\circ(s,x)(U_{jk}*U_{jk}*U_{jk}+3U_{jj}*U_{kk}*U_{jk})(s,t)\} + I_2', \qquad (4.33)
\end{aligned}
$$

where

$$
I_2' = \frac{\sqrt{s}}{2\sqrt{nb_n}} \sum_{j=1}^n (\kappa_{3,jj}-\kappa_3)\mathbb{E}\{D_{jk}^2(s)\Phi_n\}.
$$

Recall $\kappa_{3,jj}$ is the third cumulant of the jth diagonal entrie and κ_3 is the third cumulant of the off-diagonal entries. Note that

$$
|D_{jk}^l(s)\Phi_n| \le C_l(\sqrt{b_n/n}x,t), 0\le l\le 4, \qquad (4.34)
$$

So

$$
|I_2'| \le \frac{\sqrt{n}}{b_n} C_2(\sqrt{b_n/n}x,t).
$$

Consider two types of the sums above:

$$
I_{21} = \sum_{(j,k)\in I_n} U_{jj}(s,t_1)U_{jk}(s,t_2)U_{kk}(s,t_3), \qquad (4.35)
$$

$$
I_{22} = \sum_{(j,k)\in I_n} U_{jk}(s,t_1)U_{jk}(s,t_2)U_{jk}(s,t_3). \qquad (4.36)
$$

It follows from the Cauchy-Schwarz inequality that

$$
|I_{22}| \le \left(\sum_{j,k=1}^n |U_{jk}(s,t_1)|^2\right)^{1/2} \left(\sum_{j,k=1}^n |U_{jk}(s,t_2)|^2\right)^{1/2} = n. \qquad (4.37)
$$

In addition,

$$
I_{21} = n(U^{(B)}(s,t_2)V(t_1),V(t_3)), V(t) = n^{-1/2}(U_{11}(t),...,U_{nn}(t))^t. \qquad (4.38)
$$

Since $\|U^{(B)}\| \le C\ln b_n,\ \|V(t)\| \le 1$, we have $|I_{21}| \le Cn\ln b_n$. Therefore,

$$
|I_2| \le C_1\frac{x^2}{\sqrt{n}} + C_2\frac{|x|\ln b_n}{\sqrt{n}} + C_3\frac{\sqrt{n}\ln b_n}{b_n} + \frac{\sqrt{n}}{b_n}C_2(\sqrt{b_n/n}x,t).
$$

Since $\frac{\sqrt{n}\ln n}{b_n} \to 0$, then $I_2 \to 0$ on any bounded subset of $\{(x,t)|t\ge 0\}$.

Recall that $\kappa_{4,jk}=0,\ j\ne k$. Thus,

$$
I_3 = \frac{s}{3!n^{1/2}b_n^{3/2}} \sum_{j=1}^n \kappa_{4,j}\mathbb{E}\{D_{jj}^3(s)\Phi_n\}, \qquad (4.39)
$$

and

$$
|I_3| \le \frac{\sqrt{n}}{b_n^{3/2}} C_3(\sqrt{b_n/n}x,t). \qquad (4.40)
$$

Taking into account that

$$|\varepsilon_3| \leq \frac{\sqrt{n}}{b_n} C_4(\sqrt{b_n/n}x, t), \tag{4.41}$$

we conclude that $I_2, I_3, \varepsilon \to 0$ on any bounded subset of $\{(x,t) : t \geq 0\}$. It then follows from (4.30) and (4.29), that R_n, defined in (4.22), converges to 0 as $n \to \infty$. Theorem 2.2 is proven. $\square$

5 Appendix

A Poincaré Inequality

Definition A.1. A probability measure P on $\mathbb{R}^M$ satisfies the Poincaré Inequality (PI) with constant $m > 0$ if, for all continuously differentiable functions f,

$$Var_P(f) := E_P(|f(x) - E_P(f(x))|^2) \leq \frac{1}{m} E_P(|\nabla f|^2). \tag{A.1}$$

We note that the Poincaré inequality tensorises and the probability measures satisfying the Poincaré inequality have sub-exponential tails (see e.g. [1]). In particular, if P satisfies the PI on $\mathbb{R}^M$ with constant m, then for any Lipschitz continuous function G, and $|t| \leq \sqrt{m}/\sqrt{2}|G|_{\mathscr{L}}$, we have

$$E_P(e^{t(G - E_P(G))}) \leq K, \tag{A.2}$$

with $K = -\sum_{i \geq 0} 2^i \log(1 - 2^{-1}4^{-i})$. Consequently, for all $\delta > 0$,

$$P(|G - E_P(G)| \geq \delta) \leq 2K e^{-\frac{\sqrt{m}}{\sqrt{2}|G|_{\mathscr{L}}}\delta}. \tag{A.3}$$

B Decoupling formula

Definition B.1. Let ξ be a random variable such that $\mathbb{E}\{|\xi|^{p+2}\} < \infty$ for a certain nonnegative integer p. Then for any function $f : \mathbb{R} \to \mathbb{C}$ of the class C^{p+1} with bounded derivatives $f^{(l)}, l = 1, ..., p+1$, we have

$$\mathbb{E}\{\xi f(\xi)\} = \sum_{l=0}^{p} \frac{\kappa_{l+1}}{l!} \mathbb{E}\{f^{(l)}(\xi)\} + \varepsilon_p. \tag{B.1}$$

where κ_l denotes the lth cumulant of ξ and the remainder term ε_p admits the bound

$$|\varepsilon_p| \leq C_p \mathbb{E}\{|\xi|^{p+2}\} \sup_{t \in \mathbb{R}} f^{(p+1)}(t), \; C_p \leq \frac{1 + (3+2p)^{p+2}}{(p+1)!}. \tag{B.2}$$

If ξ is a Gaussian random variable with zero mean,

$$\mathbb{E}\{\xi f(\xi)\} = \mathbb{E}\{\xi^2\}\mathbb{E}\{f'(\xi)\}. \tag{B.3}$$

C Proof of Proposition 3.5

The goal of this section is to derive a bound

$$\sup_{j \neq k} |\mathbb{E}\{U_{jk}(t)\}| = O(\frac{1+t^6}{b_n}). \tag{C.1}$$

To achieve this, we first bound the mathematical expectation of the off-diagonal entries of the resolvent matrix. Then, we use the Helffer-Sjóstrand functional calculus to extend the bound to the off-diagonal entries of the unitary matrix $U(t)$.

Consider $R(z) = (z - M)^{-1}$, $Im(z) \neq 0$. The main part of the proof of proposition is the following lemma.

Lemma C.1. *Let* $|Imz| \leq 2$. *Then*

$$|\mathbb{E}\{R_{ps}\}| \leq \frac{C}{|\,\mathfrak{Im}\, z|^5 b_n}, \tag{C.2}$$

where $C > 0$ *is a constant independent from* $p \neq s$ *and* n.

Proof. We start with the resolvent identity

$$zR(z) = I + MR(z) \tag{C.3}$$

Therefore, the off-diagonal entries of $R(z)$ satisfy the following equation

$$z\mathbb{E}\{R_{ps}\} = \sum_{j:(j,p)\in I_n} \mathbb{E}\{M_{pj}R_{js}\},\ p \neq s. \tag{C.4}$$

Applying the decoupling formula, we obtain

$$\mathbb{E}\{M_{pj}R_{js}\} = \left\{ \begin{array}{ll} \frac{\sigma^2}{b_n}\mathbb{E}\{R_{jp}R_{js} + R_{jj}R_{ps}\} + \frac{\mu_3}{b_n^{3/2}}\mathbb{E}\{2R_{jp}^2 R_{js} + 2R_{jj}R_{pp}R_{js} + 4R_{jp}R_{jj}R_{ps}\} + \varepsilon_{2,j} & j \neq p \\ \frac{2\sigma^2}{b_n}\mathbb{E}\{R_{pp}R_{ps}\} + \frac{2\mu_{3,p}}{b_n^{3/2}}\mathbb{E}\{R_{pp}^2 R_{ps}\} + \varepsilon_{2,p} & j = p, \end{array} \right. \tag{C.5}$$

where

$$|\varepsilon_{2,j}| \leq \frac{C_2 \max\{\kappa_4, \kappa_4'\}}{b_n^2} \sup_{M_{pj}\in\mathbb{R}} |\frac{\partial^3 R_{js}}{\partial M_{pj}^3}| = O(\frac{1}{b_n^2 |Imz|^4}). \tag{C.6}$$

We note that

$$|\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jp}R_{js}\}| \leq \mathbb{E}\{\sqrt{\sum_{|j-p|\leq b_n} |R_{jp}|^2}\sqrt{\sum_{|j-p|\leq b_n} \mathbb{E}\{|R_{js}|^2\}} \leq \frac{1}{|Imz|^2}, \tag{C.7}$$

$$|\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jp}^2 R_{js}\}| \leq \frac{1}{|Imz|^2} \sum_{j:|j-p|\leq b_n} \mathbb{E}\{|R_{js}|\} \leq \frac{\sqrt{2b_n+1}}{|Imz|^2}\sqrt{\sum_{j:|j-p|\leq b_n} |R_{js}|^2} \leq \frac{\sqrt{2b_n+1}}{|Imz|^3} \tag{C.8}$$

and similarly,

$$|\textstyle\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jj}R_{pp}R_{js}\}| \leq \frac{\sqrt{2b_n}}{|Imz|^3}, \quad |\displaystyle\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jp}R_{jj}R_{ps}\}| \leq \frac{\sqrt{2b_n}}{|Imz|^3}. \tag{C.9}$$

Thus, for $p \neq s$,

$$\begin{aligned} z\mathbb{E}\{R_{ps}\} &= \sum_{j:(j,p)\in I_n} \frac{\sigma^2}{b_n}\mathbb{E}\{R_{jj}R_{ps}\} + O(\frac{1}{b_n|Imz|^2}) + O(\frac{1}{b_n|Imz|^3}) + O(\frac{1}{b_n|Imz|^4}) \\ &= \sum_{j:(j,p)\in I_n} \frac{\sigma^2}{b_n}\mathbb{E}\{R_{jj}R_{ps}\} + O(\frac{1}{b_n|Imz|^4}). \end{aligned} \tag{C.10}$$

Since the diagonal entries R_{jj}'s have the same distribution, we can write $g_n(z) := \frac{1}{n}\mathbb{E}\{TrR\} = \mathbb{E}\{R_{jj}\}$. From the Wigner semicircle law for band random matrices,

$$g_n(z) \to \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{\sqrt{8\sigma^2 - x^2}}{4\pi\sigma^2(z-x)}dx. \tag{C.11}$$

We have

$$\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jj}R_{ps}\} = (2b_n+1)g_n(z)\mathbb{E}\{R_{ps}\} + \sum_{|j-p|\le b_n} \mathbb{E}\{R_{jj}^\circ R_{ps}^\circ\}, \tag{C.12}$$

and

$$|\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jj}^\circ R_{ps}^\circ\}| \le (2b_n+1)Var^{1/2}\{R_{11}\}Var^{1/2}\{R_{ps}\}. \tag{C.13}$$

The Poincaré inequality implies that

$$Var\{R_{ps}\} \le \frac{1}{mb_n}\sum_{j:(j,p)\in I_n} \mathbb{E}\{\beta_{jk}^2|R_{pj}R_{ks}+R_{pk}R_{js}|^2\} \le \frac{2}{mb_n|Imz|^4}. \tag{C.14}$$

Hence,

$$z\mathbb{E}\{R_{ps}\} = \frac{\sigma^2(2b_n+1)}{b_n}g_n(z)\mathbb{E}\{R_{ps}\} + O(\frac{1}{|\,\mathfrak{Im}\,z|^4 b_n}), \tag{C.15}$$

which implies

$$[z - \frac{2b_n+1}{b_n}\sigma^2 g_n(z)]\mathbb{E}\{R_{ps}\} = O(\frac{1}{|\,\mathfrak{Im}\,z|^4 b_n}). \tag{C.16}$$

In a similar fashion,

$$[z - \frac{2b_n+1}{b_n}\sigma^2 g_n(z)]g_n(z) = 1 + O(\frac{1}{|\,\mathfrak{Im}\,z|^4 b_n}). \tag{C.17}$$

If the term $O(\frac{1}{|\,\mathfrak{Im}\,z|^4 b_n})$ at the r.h.s. of (C.17) is bounded in absolute value from above by 1/2, then there exists a constant C_1 such that

$$\frac{C_1}{b_n|Imz|^4} \le 1/2. \tag{C.18}$$

Then

$$|[z - \frac{2b_n+1}{b_n}\sigma^2 g_n(z)]g_n(z)| \ge 1/2, \tag{C.19}$$

$$|z - \frac{2b_n+1}{b_n}\sigma^2 g_n(z)| \ge \frac{1}{2|g_n(z)|} \ge \frac{|Imz|}{2}, \tag{C.20}$$

and (C.20) and (C.16) imply

$$|\mathbb{E}\{R_{ps}\}| = O(\frac{1}{|\,\mathfrak{Im}\,z|^5 b_n}). \tag{C.21}$$

Now assume that

$$\frac{C_1}{b_n|Imz|^4} > 1/2. \tag{C.22}$$

Then

$$|\mathbb{E}\{R_{ps}\}| \le \frac{1}{|Imz|} < \frac{2C_1}{b_n|Imz|^5}. \tag{C.23}$$

Lemma C.1 is proven. □

Now, we extend the bound in the last lemma to the off-diagonal entries of $f(M)$, where f is sufficiently smooth function with compact support. To this end, we use the Helffer-Sjöstrand functional calculus (see e.g. [17], [30]). We write

$$\mathbb{E}\{f(M)_{jk}\} = -\mathbb{E}\left\{\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} R_{jk}dxdy\right\} = -\frac{1}{\pi}\int_{\mathbb{R}\times[-1,1]} \frac{\partial \tilde{f}}{\partial \bar{z}} O(\frac{1}{|\,\mathfrak{Im}\,z|^5 b_n})dxdy, \tag{C.24}$$

where

i) $z = x + iy$ with $x, y \in \mathbb{R}$;

ii) $\tilde{f}(z)$ is the extension of the function f defined as

$$\tilde{f}(z) := \Big(\sum_{n=0}^{l} \frac{f^{(n)}(x)(iy)^n}{n!} \Big) \sigma(y); \tag{C.25}$$

here $\sigma \in C^\infty(\mathbb{R})$ is a nonnegative function equal to 1 for $|y| \le 1/2$ and equal to zero for $|y| \ge 1$.

Since

$$\frac{\partial \tilde{f}}{\partial \bar{z}} := \frac{1}{2}\Big(\frac{\partial \tilde{f}}{\partial x} + i \frac{\partial \tilde{f}}{\partial y} \Big), \tag{C.26}$$

one has (with $l = 5$)

$$\frac{\partial \tilde{f}}{\partial \bar{z}} = \frac{1}{2} \left(\sum_{n=0}^{5} \frac{f^{(n)}(x)(iy)^n}{n!} \right) i \frac{d\sigma}{dy} + \frac{1}{2} f^{(6)}(x)(iy)^5 \frac{\sigma(y)}{5!}. \tag{C.27}$$

In particular,

$$\Big| \frac{\partial \tilde{f}}{\partial \bar{z}} \Big| \le const \|f\|_{C_c^6(\mathbb{R})} |y|^5, \tag{C.28}$$

for six times continuously differentiable function f with compact support, where

$$\|f\|_{C_c^6(\mathbb{R})} = \max_{0 \le k \le 6} \max_{x \in \mathbb{R}} |f^{(k)}(x)|. \tag{C.29}$$

Combining (C.23), (C.24), and (C.29), we arrive at

$$\mathbb{E}\{f(M)_{jk}\} = O(\frac{\|f\|_{C_c^6(\mathbb{R})}}{b_n}). \tag{C.30}$$

This bound is not sufficient for our purposes since $g(x) = e^{itx}$ is not compactly supported. Let $f(x) \in C^\infty(\mathbb{R})$ be a function satisfying $f(x) \equiv g(x)$ if $x \in [-10\sigma, 10\sigma]$, $f(x) = 0$ if $|x| > 20\sigma$. If $Spec(M) \subset [-10\sigma, 10\sigma]$, we clearly have $f(M) = g(M)$. Hence,

$$|\mathbb{E}\{g(M)_{jk}\}| \le |\mathbb{E}\{f(M)_{jk}\}| + \sup_{x \in \mathbb{R}} |g(x)| \mathbb{P}(\|M\| \ge 10\sigma). \tag{C.31}$$

In the next lemma, we show that $\mathbb{P}(\|M\| \ge 10\sigma)$ is negligibly small.

Lemma C.2. *There exists a positive constant C such that*

$$\mathbb{P}(\|M\| \ge 10\sigma) \le C e^{-C\sqrt{b_n}\sigma}. \tag{C.32}$$

Clearly, (C.31) and (C.2) finish the proof of Proposition 3.5. Thus, we are left with proving (C.32).

Proof. We note that $\|M\|$ is a Lipschitz function of the matrix entries and the distribution of the entries of M satisfies the Poincaré inequality. Therefore, we have

$$\mathbb{P}(|\|M\| - \mathbb{E}\{\|M\|\}| \ge \delta) \le c_1 e^{-c_2\sqrt{b_n}\delta}, \tag{C.33}$$

with some positive constants c_1 and c_2. Below we show that $\mathbb{E}\{\|M\|\} \le 5\sigma$ for all sufficiently large n.

Let $\tilde{M}$ be an independent copy of M. Using a symmetrization argument (see e.g. [42]), we have

$$\mathbb{E}\{\|M - \tilde{M}\|\} \ge \mathbb{E}\{\|M\|\} \tag{C.34}$$

Denote $B = M - \tilde{M}$. Applying the method of moments ([35], [36]), one can show that

$$\mathbb{E}\{TrB^{2s}\} = \frac{(16\sigma^2)^s n}{\sqrt{\pi s^3}}(1 + o(1)), \tag{C.35}$$

as $n \to \infty$ provided $s \to \infty$ so that $s = o(b_n^{1/3})$. The computations are standard and left to the reader. Then

$$\mathbb{E}\{\|B\|^{2s}\} \le \frac{(16\sigma^2)^s n}{\sqrt{\pi s^3}}(1 + o(1)), \tag{C.36}$$

which implies

$$\mathbb{E}\{\|B\|\} \le 4\sigma \left[\frac{n}{\sqrt{\pi s^3}}(1 + o(1))\right]^{1/2s}. \tag{C.37}$$

Therefore, for sufficiently large n,

$$\mathbb{E}\{\|M\|\} \le \mathbb{E}\{\|B\|\} \le 5\sigma. \tag{C.38}$$

The last inequality and (C.33) finish the proof of Lemma C.2 and Proposition 3.5. □

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